Investigation of Spectrum for a Sturm–Liouville problem with Two-Point Nonlocal Boundary Conditions

Academic supervisor: Prof. dr. Artūras Štikonas

Kristina Bingelė

Vilnius University, Vilnius, LITHUANIA

Important problems

- J.R. Cannon (1963),
  "The solution of the heat equation subject to specification of energy";
- L.I. Kamynin (1964),
  "A boundary value problem in the theory of the heat conduction with nonclassical boundary condition";
- A.V. Bitsadze and A.A. Samarskii (1969),
  "Some elementary generalizations of linear elliptic boundary value problems";
- N.I. Ionkin (1979),
  "A problem for the heat equation with a nonclassical (nonlocal) boundary condition";
- G. Infante (2003),
  "Eigenvalues of some non-local boundary-value problems.";
- A. Ashyralyev (2008),
  "A note on the Bitsadze–Samarskii type nonlocal boundary value problem in a Banach space.";

- R. Čiegis (1984), *"The numerical solution of the heat equation with nonclassical condition"*;
- R. Čiegis (1988), *"Numerical solution of a problem with small parameter for the highest derivative and a nonlocal condition"*;
- R. Čiegis, O. Štikonienė, A. Štikonas and O. Subač (2001), *"Stationary problems with nonlocal boundary conditions"*;
- R. Čiupaila, Ž. Jesevičiūtė and M. Sapagovas (2004), *"On the eigenvalue problem for one-dimensional differential operator with nonlocal integral condition"*;
- M.P. Sapagovas and A.D. Štikonas (2005), *"On the structure of the spectrum of a differential operator with a nonlocal condition"*;
- M.P. Sapagovas (2004), *"Diferencialiniu lygčiu kraštiniai uždaviniai su nelokaliosiomis salygomis"*;
- T. Leonavičienė, A. Bugajev, G. Jankevičiūtė and R. Čiegis (2016), *"On stability analysis of finite difference schemes for generalized Kuramoto–Tsuzuki equation with nonlocal boundary conditions"*;

Important problems

Sturm–Liouville Problem with Nonlocal Boundary Conditions (NBCs):

- Are important for investigation of the existence and uniqueness of stationary problems soliution [Ionkin 1996, Gulin 2003]
- Very complicated because are not self-adjoint
- Spectrum for such problems may be not positive (or real).
- Useful for investigation the stability of the finite difference schemes for nonstationary problems
- Useful for investigation of the convergence of iterative metods

- S. Pečiulytė and A. Štikonas (2006), *"Sturm–Liouville Problem for Stationary Differential Operator with Nonlocal Two-Point Boundary Conditions"*;
- S. Pečiulytė and A. Štikonas (2007), *"Distribution of the critical and other points in boundary problems with nonlocal boundary condition"*;
- S. Pečiulytė, O. Štikonienė and A. Štikonas (2008), *"Investigation of Negative Critical Points of the Characteristic Function for Problems with Nonlocal Boundary Conditions"*;
- A. Štikonas and O. Štikonienė (2009), *"Characteristic Functions for Sturm–Liouville Problems with Nonlocal Boundary Conditions"*;
- A. Skučaitė and A. Štikonas. (2015), *"Spectrum Curves for Sturm—Liouville Problem with Integral Boundary Condition"*;
- A. Skučaitė and A. Štikonas. (2016), *"Investigation of the spectrum for Sturm–Liouville problem with a nonlocal integral condition"*;

Aims and problems

In the case of the differential Sturm–Liouville Problem

$$-u'' = \lambda u, \quad t \in (0,1), \tag{1}$$
$$u(0) = 0, \quad \text{or} \quad u'(0) = 0 \tag{2}$$

we investigate the following NBC:

$$u(1) = \gamma u(\xi), \qquad u(1) = \gamma u'(\xi),$$
$$u'(1) = \gamma u(\xi), \qquad u'(1) = \gamma u'(\xi),$$
$$u(\xi) = \gamma u(1-\xi),$$

where $\gamma \in \mathbb{R}$ ir $\xi \in [0,1]$.

Main problems:

- find Constant Eigenvalues, which do not depend on parameter γ;
- find Zeroes, Poles and Critical Points of Characteristic Function;
- describe Spectrum Curves and investigate their properties;
- investigate the dependence of Spectrum Domain on parameter ξ in NBC, find bifurcation points and types.

Aims and problems

In the case of discrete Sturm–Liouville Problem we approximate differential equation by the following Finite-Difference Scheme (FDS)

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}. \tag{3}$$

and investigate the following NBC:

$$U_n = \gamma \frac{U_{m+1} - U_{m-1}}{2h}, \qquad\qquad U_n = \gamma U_m.$$

At the left side of interval one of the conditions was selected:

$$U_0 = 0, \qquad\qquad U_1 = U_0.$$

The discrete problem was obtained by approximating the differential problem by a finite difference scheme.

Main problems:

- find Constant Eigenvalues, which do not depend on parameter γ;
- find Zeroes, Poles and Critical Points of Characteristic Function;
- determine the dependence of these points on the number of grid points;
- investigate the behavior of Spectral Curves in the neighborhood of special special points;
- find the quantitative relationships between the numbers of points mentioned.

SLP with one classical BC and another two-point NBC

$$-u'' = \lambda u, \quad t \in (0,1), \tag{4}$$
$$u(0) = 0, \tag{5}$$
$$u(1) = \gamma u'(\xi), \tag{6}$$

where parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$. The eigenvalue $\lambda \in \mathbb{C}_\lambda := \mathbb{C}$ and eigenfunction $u(t)$ can be complex function.
If $\gamma = 0$, then we have the SLP with classical BCs. In this case eigenvalues and eigenfunction are known:

$$\lambda_k = (k\pi)^2, \qquad u_k(t) = \sin(k\pi t),\ k \in \mathbb{N} \tag{7}$$

The case $\gamma = \infty$ corresponds to (4) with clasical BCs $u(0) = 0$ and $u'(\xi) = 0$, $\xi \in [0,1]$, instead of condition (6) and eigenvalues and eigenfunction are:

$$\lambda_k = \big((k-1/2)\pi/\xi\big)^2, \qquad u_k(t) = \sin\big((k-1/2)\pi t/\xi\big),\ k \in \mathbb{N}. \tag{8}$$

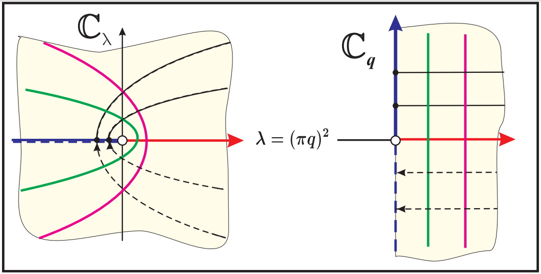


Figure 1: Bijective mapping $\lambda = (\pi q)^2$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q$ [**?**].

Nontrivial solution of the problem (4)–(6) exists if $q \in \mathbb{C}_q$ is the root of a equation

$$\frac{\sin(\pi q)}{\pi q} = \gamma \cos(\xi \pi q) \quad Z(q) = \gamma P_\xi(q). \tag{9}$$

We will define a *Constant Eigenvalue* (CE) as the eigenvalue which does not depend on the parameter γ. Then for any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system:

$$Z(q) = 0, \qquad P_\xi(q) = 0, \tag{10}$$

$\lambda = 0$

Eigenvalue $\lambda = 0$ exists if and only if $\gamma = 1$.

Lemma 1.6.

For SLP (4)–(6) Constant Eigenvalues exist only for rational parameter $\xi = m/n \in (0,\ 1)$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$, values and those eigenvalues are equal to $\lambda_s = (\pi c_s)^2$, $c_s := (s - 1/2)n$, $s \in \mathbb{N}$.

For SLP (4)–(6) we have meromorphic Complex Characteristic Functions (Complex CF):

$$\gamma_c(q) = \gamma_c(q; \xi) := \frac{Z(q)}{P_\xi(q)} = \frac{\sin(\pi q)}{\cos(\xi \pi q)}, \quad z \in \mathbb{C}, \tag{11}$$

γ-points of Complex CF define EPs (and Eigenvalues, too) which depend on parameter γ. We call such EPs *Nonconstant Eigenvalue's Points* (NEP) and corresponding Eigenvalues as *Nonconstant Eigenvalues*.

Remark 1.1.

If the parameter $\xi = 0$, then from the formula (9) we obtain that $P_\xi \equiv 1$. So, $\overline{\mathcal{Z}}_\xi = \varnothing$ and CEPs do not exist. If $\xi = 1$ then there are no CEPs, because the functions $\sin(\pi q)$ and $\cos(\pi q)$ have no common zeroes (we have the third type BC).

Remark 1.2.

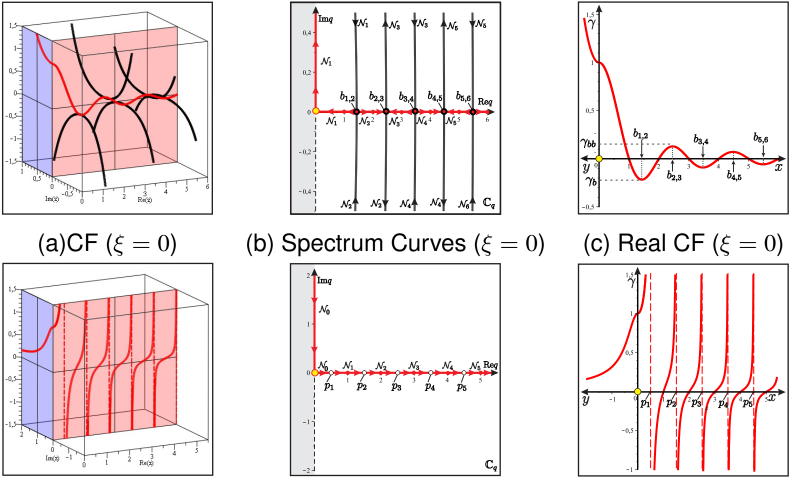
If the parameter $\xi \notin \mathbb{Q}$, then CEPs do not exist, because the equation $\xi l = k - \frac{1}{2}$ has not roots for $l, k \in \mathbb{N}$.

Remark 1.3.

If $\xi \in \mathbb{Q}$, $\xi = m/n$ and $n \in \mathbb{N}_e$ then the right hand side of equation $nk - lm = \frac{n}{2}$ is integer number. If $n \in \mathbb{N}_o$ then this equation has no roots.

Remark 1.7.

In the case $\xi = 0$ function $P_\xi \equiv 1$ and PPs do not exist. If $\xi > 0$ a set of poles $\mathcal{P}_\xi = \varnothing$ or countable. So, PPs exist if $\xi \neq 1/n$.



(a)CF ($\xi = 0$) (b) Spectrum Curves ($\xi = 0$) (c) Real CF ($\xi = 0$)

(d)CF ($\xi = 1$) (e) Spectrum Curves ($\xi = 1$) (f) Real CF ($\xi = 1$)

Figure 2: CCF, Spectrum Domain, Real CF for $\xi = 0$, $\xi = 1$. ● – Zero Point, ○ – Pole Point, ● – Ramification Point, ● – Branch Eigenvalue Point, ● – Critical Point.

Real Characteristic Function (Real CF) describes only real Nonconstant Eigenvalues and it is restriction of the Complex CF $\gamma_c(q)$ on the set $\mathbb{R}_q$. We can use the argument $x \in \mathbb{R}$ for Real CF:

$$\gamma_r(x) = \gamma_r(x;\xi) := \begin{cases} \gamma(-\imath x;\xi) = \frac{\sinh(\pi x)}{\pi x \cosh(\xi \pi x)}, & x \leqslant 0; \\ \gamma(x;\xi) = \frac{\sin(\pi x)}{\pi x \cos(\xi \pi x)}, & x \geqslant 0. \end{cases} \tag{12}$$

This function is useful for investigation of real negative, zero and positive eigenvalues

$$\lambda = \lambda_r(x) = \lambda_r(x;\xi) := \begin{cases} -(\pi x)^2, & x \leqslant 0; \\ (\pi x)^2, & x \geqslant 0. \end{cases} \tag{13}$$

Remark 1.11.

In the case $\xi = \xi_c = \frac{1}{\sqrt{3}}$ the point $q = 0$ is 3CP in the domain $\mathbb{C}_q$, but for $\lambda = 0$ it is only 1CP, because $q = 0$ is RP for map $\lambda = (\pi q)^2$. In the complex plane $\mathbb{C}_\lambda$ the Taylor series CF $\gamma(q)$ have a form

$$\gamma(\lambda,\xi) := 1 + \left(-\tfrac{1}{6} + \tfrac{1}{2}\xi^2\right)\lambda + \left(\tfrac{1}{120} - \tfrac{1}{24}\xi^4 - \left(\tfrac{1}{2}\left(\tfrac{1}{6} - \tfrac{1}{2}\xi^2\right)\right)\xi^2\right)\lambda^2 + \mathcal{O}(\lambda^3). \tag{14}$$

If $\xi \neq \xi_c$, then point $q = 0$ and $\lambda = 0$ are not CPs.

Lemma 1.12.

Zero Point of CF can not be CP.

Remark 1.13.

Pole Point of CF is not CP. Function γ^{-1} has CP at this point only if order of the pole is greater than the first

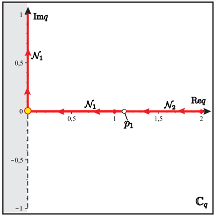
Remark 1.14.

In the case $\xi = 1$ and $\gamma \neq 0$ we can consider boundary condition $u'(1) = \tilde{\gamma}u(1)$, $\tilde{\gamma} \in \mathbb{R}$, where $\tilde{\gamma} = \gamma^{-1}$. Now CF is $\tilde{\gamma} = \pi q \cos(\pi q)/\sin(\pi q)$ and its zeroes are $\tilde{z}_k = p_k$, $k \in \mathbb{N}$, poles are $\tilde{p}_k = z_k$, $k \in \mathbb{N}$ (for $\xi = 1$ CEPs do not exist, but in the general case $\tilde{c}_k = c_k$ for all k). For parameter $\tilde{\gamma} \in \mathbb{R}$ all Spectrum Curves will be regular.

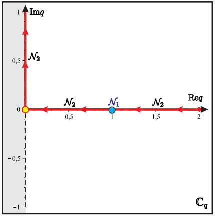
Remark 1.9.

A point $q = \infty \notin \mathbb{C}_q$. This point is singular (isolated essential point if $\mathcal{P}_\xi = \varnothing$, otherwise we have cluster of poles) point.

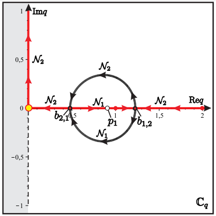
Zero and Pole bifurcation type β_{ZP}



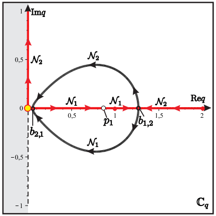
(a) $\xi = 0.45$



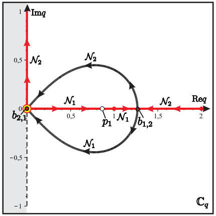
(b) $\xi = \frac{1}{2}$



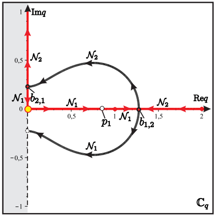
(c) $\xi = 0.55$



(d) $\xi = 0.577$



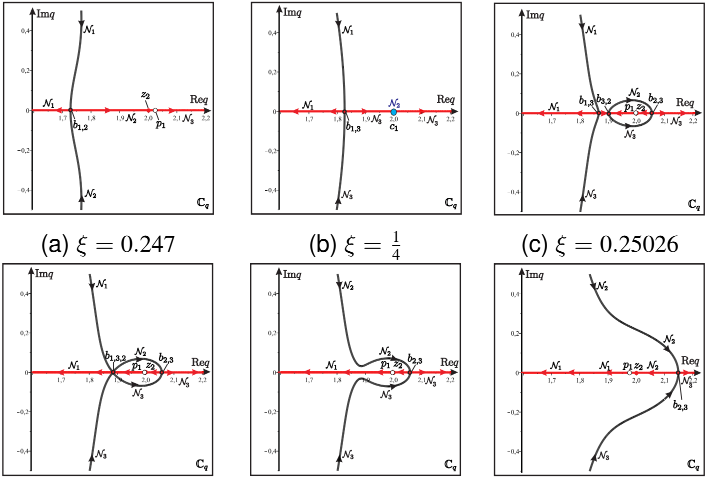
(e) $\xi = \xi_c = \frac{1}{\sqrt{3}}$



(f) $\xi = 0.584$

Figure 3: Spectrum Curves for various parameter ξ values.
– Critical Point at Branch Eigenvalue Point.

The second order CP bifurcation β_{2B}

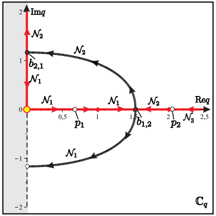


(a) $\xi = 0.247$ (b) $\xi = \frac{1}{4}$ (c) $\xi = 0.25026$

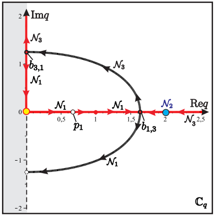
(d) $\xi = \xi_{2b} \approx 0.25028$ (e) $\xi = 0.2503$ (f) $\xi = 0.253$

Figure 4: Spectrum Curves for various parameter ξ values, bifurcations.

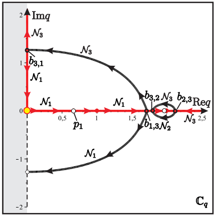
The second order CP bifurcation β_{2B}



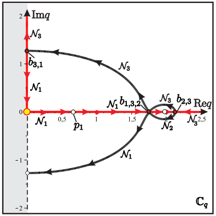
(a) $\xi = 0.735$



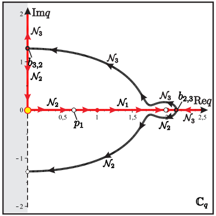
(b) $\xi = \frac{3}{4}$



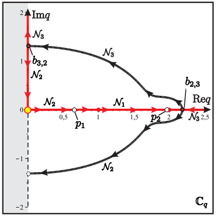
(c) $\xi = 0.7535$



(d) $\xi = \xi_{2b} \approx 0.7539...$



(e) $\xi = 0.7545$



(f) $\xi = 0.76$

Figure 5: Spectrum Curves for various parameter ξ values, bifurcations.

Conclusions of this Chapter

1. For SLP (4)–(6) CEs do not exist for irrational parameter ξ and exist only for $\xi = \frac{m}{n} \in \mathbb{Q}$, $0 < m < n$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$.

2. SLP (4)–(6) has two types CPs: the first, the second order. We have only one 3CP, $b_{2,1} = 0$, $\xi = \xi_c = 1/\sqrt{3}$. But this point is 1CP in the domain $\mathbb{C}_\lambda$. The negative CP exists only for $\xi > \xi_c$.

3. For SLP (4)–(6) we obtain two types' bifurcations:
   - $\beta_{ZP}: (z_{l_s}, p_{k_s}) \to c_s \to (b_{l_s+1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$
     when zero and pole of CF merge into CEP and we get a loop type curve.
   - $\beta_{2B}: (b_{l_s-1,l_s+1}, b_{l_s+1,l_s}) \to b_{l_s-1,l_s+1,l_s} \to \varnothing$
     when two 1CPs merge into one 2CP. At this bifurcation the loop type curve vanish.

Let us analyze SLP with one classical BC

$$-u'' = \lambda u, \quad t \in (0,1), \tag{15}$$
$$u(0) = 0, \tag{16}$$

and another two-point NBC of Samarskii–Bitsadze type:

$$u'(1) = \gamma u(\xi), \qquad \text{Case 1} \tag{17a}$$
$$u'(1) = \gamma u'(\xi), \qquad \text{Case 2} \tag{17b}$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$.

S. Pečiulytė and A. Štikonas, 2005–2008.

If $\gamma = 0$, we have problems with classical BCs. In this case, all the eigenvalues are positive and eigenfunctions do not depend on the parameter ξ:

$$\lambda_k = \pi^2(k-1/2)^2, \quad u_k = \sin(\pi(k-1/2)t), \quad k \in \mathbb{N}. \tag{18}$$

There exists a nontrivial solution (eigenfunction) if q is the root of the function:

$$\cos(\pi q) = \gamma \frac{\sin(\xi \pi q)}{\pi q}, \qquad Z(q) = \gamma P_\xi(q) \quad q \in \mathbb{C}, \tag{19a}$$

$$\cos(\pi q) = \gamma \cos(\xi \pi q), \qquad Z(q) = \gamma P_\xi(q) \quad q \in \mathbb{C}. \tag{19b}$$

Lemma 2.3.

For SLP (15)–(17a) Constant Eigenvalues exist only for rational parameter $\xi = m/n \in (0,\, 1)$, $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$, values and those eigenvalues are equal to $\lambda_s = (\pi c_s)^2$, $c_s := (s - 1/2)n$, $s \in \mathbb{N}$.

Lemma 2.4.

For SLP (15)–(17b) Constant Eigenvalues exist only for rational parameter $\xi = m/n \in (0,\, 1)$, $m, n \in \mathbb{N}_o$, values and those eigenvalues are equal to $\lambda_s = (\pi c_s)^2$, $c_s := n(s - 1/2)$, $s \in \mathbb{N}$.

For SLP (4)–(6) we have meromorphic Complex Characteristic Functions (Complex CF)

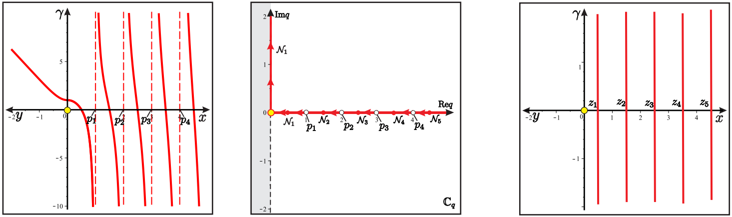
$$\gamma_c(q) = \frac{Z(q)}{P_\xi(q)} = \frac{\pi q \cos(\pi q)}{\sin(\xi \pi q)}, \quad q \in \mathbb{C}_q, \tag{20a}$$

$$\gamma_c(q) = \frac{Z(q)}{P_\xi(q)} = \frac{\cos(\pi q)}{\cos(\xi \pi q)}, \quad q \in \mathbb{C}_q. \tag{20b}$$

Real Characteristic Function (Real CF) describes only real Nonconstant Eigenvalues and it is restriction of the Complex CF $\gamma_c(q)$ on the set $\mathbb{R}_q$:

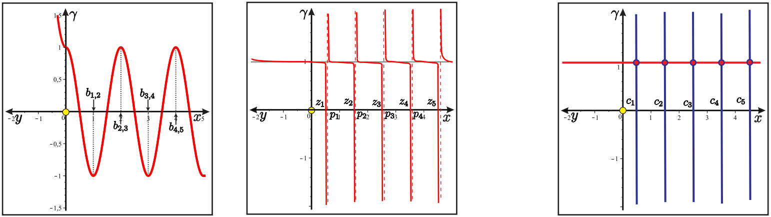
$$\gamma_r(x) = \gamma_r(x;\xi) = \begin{cases} \frac{\pi x \cosh(\pi x)}{\sinh(\xi \pi x)}, & x \leqslant 0; \\ \frac{\pi x \cos(\pi x)}{\sin(\xi \pi x)}, & x \geqslant 0. \end{cases} \tag{21a}$$

$$\gamma_r(x) = \gamma_r(x;\xi) = \begin{cases} \frac{\cosh(\pi x)}{\cosh(\xi \pi x)}, & x \leqslant 0; \\ \frac{\cos(\pi x)}{\cos(\xi \pi x)}, & x \geqslant 0. \end{cases} \tag{21b}$$



(a) Real CF (b) Spectrum Curves (c) $\xi = 0$ (the limit case)

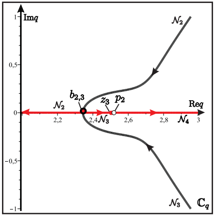
Figure 6: Real CF and Spectrum Curves for $\xi = 1$ and Real CF for $\xi = 0$ in Case 1.



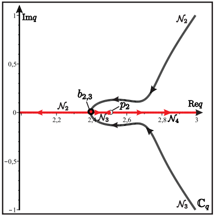
(a) Real CF $\xi = 0$ (b) Real CF $\xi = \frac{499}{500}$ (c) Real CF $\xi = 1$ (the limit case)

Figure 7: Real CF for $\xi = 0$ and $\xi \lesssim 1$ in Case 2.

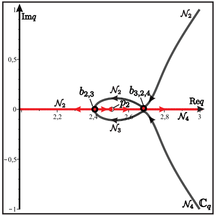
Bifurcations β_{ZP}^{-1} and β_{2B}^{-1} in Case 1.



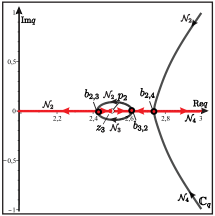
(a) $\xi = 0.795$



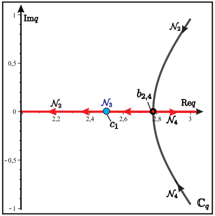
(b) $\xi = 0.798$



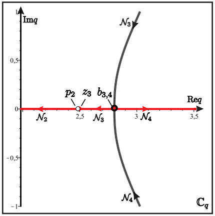
(c) $\xi = \xi_{2b} = 0.798552\ldots$



(d) $\xi = 0.799$



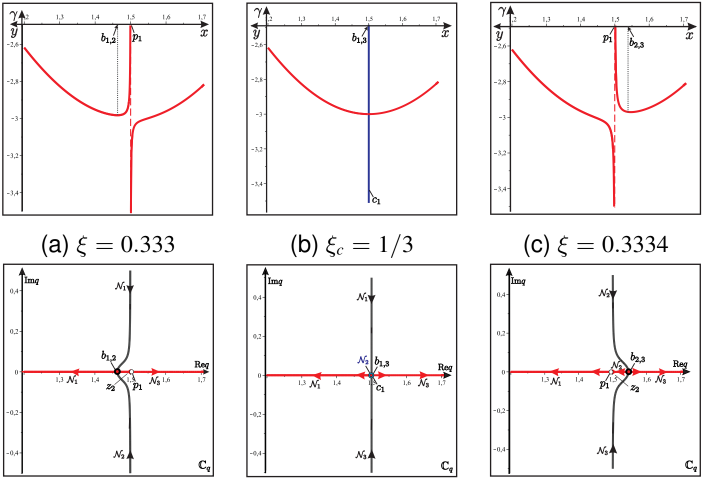
(e) $\xi = \xi_c = 4/5$



(f) $\xi = 0.801$

Figure 8: Spectrum Curves for various parameter ξ values in Case 1

Symmetric Zero and Pole bifurcation β^0_{ZP} in Case 2.



(a) $\xi = 0.333$ (b) $\xi_c = 1/3$ (c) $\xi = 0.3334$

(d) $\xi = 0.333$ (e) $\xi_c = 1/3$ (f) $\xi = 0.3334$

Figure 9: CF and bifurcation in Case 2.

Let us investigate SLP

$$-u'' = \lambda u, \quad t \in (0,1), \tag{22}$$

with one classical (Neumann type) BC:

$$u'(0) = 0, \tag{23}$$

and another two-point NBC $(0 \leqslant \xi \leqslant 1)$:

$$u'(1) = \gamma u(\xi), \tag{24a}$$

$$u'(1) = \gamma u'(\xi), \tag{24b}$$

$$u(1) = \gamma u'(\xi), \tag{24c}$$

$$u(1) = \gamma u(\xi), \tag{24d}$$

where parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$.
If $\gamma = 0$, we obtain classical BVP. In this case, all the eigenvalues are positive and eigenfunctions do not depend on the parameter ξ:

$$\lambda_k = (\pi k)^2, \qquad u_k = \cos(\pi k t), \qquad k \in \mathbb{N} \quad (\text{ Case 1 and 2}), \tag{25a}$$

$$\lambda_k = \pi^2 (k - 1/2)^2, \qquad u_k = \cos(\pi (k - 1/2) t), \qquad k \in \mathbb{N} \quad (\text{ Case 3 and 4}). \tag{25b}$$

Theorem 2.9.

Spectra for SLPs (15)–(17b) and (22)–(24d) overlap for all γ and ξ.

$\lambda = 0$

Eigenvalue $\lambda = 0$ ($C \neq 0$) exists if and only if:

- $\gamma = 0$ in Case 1;
- γ is any number in Case 2;
- $\gamma = 1$ in Case 4.

In Case 3 eigenvalue $\lambda = 0$ does not exist.

There exists a nontrivial solution (eigenfunction) if q is the root of the function:

$$-\pi q \sin(\pi q) = \gamma \cos(\pi q \xi), \tag{26a}$$

$$q \sin(\pi q) = \gamma q \sin(\pi q \xi), \tag{26b}$$

$$-\cos(\pi q) = \gamma \pi q \sin(\pi q \xi), \tag{26c}$$

$$\cos(\pi q) = \gamma \cos(\pi q \xi). \tag{26d}$$

We see that (26d) in Case 4 is the same as (19b) in Case 2.

Kristina Skučaitė-Bingelė and Artūras Štikonas (2011);
"Investigation of complex eigenvalues for a stationary problem with two-point nonlocal boundary condition".

Artūras Štikonas and Olga Štikonienė (2009);
"Characteristic functions for Sturm–Liouville problems with nonlocal boundary conditions" (Case 2 (26b)).

We introduce two entire functions:

$$Z(q) := \pi q \sin(\pi q); \quad P_\xi(q) := -\cos(\xi \pi q), \qquad q \in \mathbb{C}, \tag{27a}$$

$$Z(q) := \pi q \sin(\pi q); \quad P_\xi(q) := \pi q \sin(\pi q \xi), \qquad q \in \mathbb{C}, \tag{27b}$$

$$Z(q) := \cos(\pi q); \quad P_\xi(q) := -\pi q \sin(\pi q \xi), \qquad q \in \mathbb{C}. \tag{27c}$$

For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system:

$$Z(q) = 0, \qquad P_\xi(q) = 0. \tag{28}$$

Corollary 2.10.

Spectrum Curves and Spectrum Domain $\mathcal{N}_\xi$ for SLPs (15)–(17b) and (22)–(24d) are the same.

Remark 2.13.

In Case 2 RP $q = 0$ is CEP (of the second order). For the other cases CEPs are positive.

Remark 2.18.

CEP at Ramification Point $c_0 = 0$ in Case 2 is double in $\mathbb{C}_q$ but corresponding CE $\lambda = 0$ is simple.

Lemma 2.15.

For SLP (22)–(24a) Constant Eigenvalues exist only for rational parameter $\xi = m/n \in (0,\ 1)$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$, $\gcd(m,n) = 1$, values and those eigenvalues are equal to $\lambda_s = (\pi c_s)^2$, $c_s := n(s - 1/2)$, $s \in \mathbb{N}$.

Lemma 2.16.

For SLP (22)–(24b) Constant Eigenvalues exist only for rational parameter $\xi = m/n \in (0,\ 1)$, $m, n \in \mathbb{N}$, $\gcd(m,n) = 1$, values and those eigenvalues are equal to $\lambda_s = (\pi c_s)^2$, $c_s := ns$, $s \in \mathbb{N}_0$.

Lemma 2.17.

For SLP (22)–(24c) Constant Eigenvalues exist only for rational parameter $\xi = m/n \in (0,\ 1)$, $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$, $\gcd(m,n) = 1$, values and those eigenvalues are equal to $\lambda_s = (\pi c_s)^2$, $c_s := n(s - 1/2)$, $s \in \mathbb{N}$.

For SLP (22)–(24) we have meromorphic Complex Characteristic Functions (Complex CF)

$$\gamma_c(q) : -\frac{\pi q \sin(\pi q)}{\cos(\xi\pi q)}, \tag{29a}$$

$$\gamma_c(q) := \frac{\sin(\pi q)}{\sin(\xi\pi q)}, \tag{29b}$$

$$\gamma_c(q) := -\frac{\cos(\pi q)}{\pi q \sin(\xi\pi q)}. \tag{29c}$$

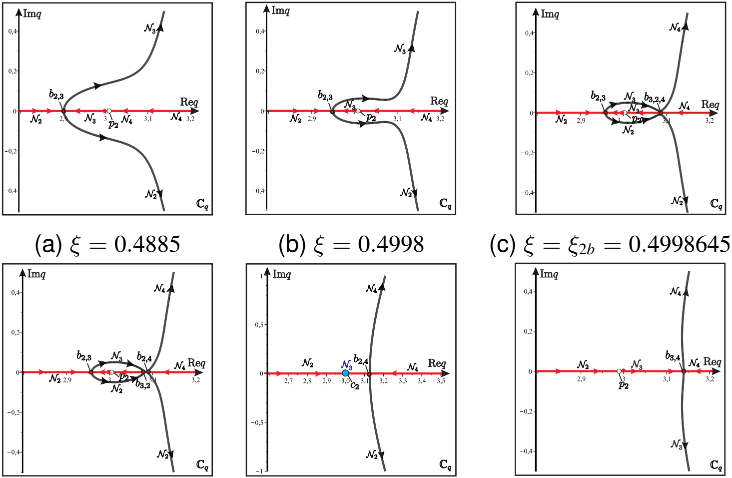
Theorem 2.25.

Spectrum of SLP (22)–(24) has one additional simple eigenvalue $\lambda = 0$ in addition to eigenvalues of spectrum of SLP (22)–(24) in Introduction [A. Štikonas and O. Štikonienė 2009] for all γ and $\xi \in (0, 1)$.

Corollary 2.26.

Additional eigenvalue $\lambda = 0$ for SLP (22)–(24b) corresponds to nonregular Spectrum Curve (CEP $q = 0$) $\mathcal{N}_0$. The other Spectrum curves for both SLPs overlap

Bifurcations β_{ZP}^{-1} and β_{2B}^{-1} in Case 1.

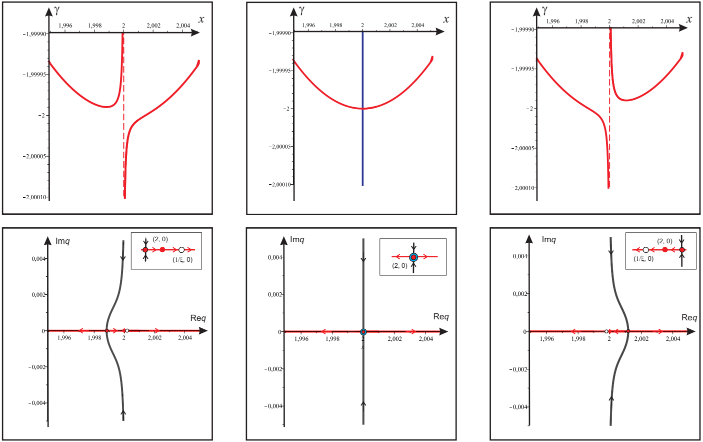


(a) $\xi = 0.4885$ (b) $\xi = 0.4998$ (c) $\xi = \xi_{2b} = 0.4998645\ldots$

(d) $\xi = 0.499867$ (e) $\xi = \xi_c = 1/2$ (f) $\xi = 0.5001$

Figure 10: Bifurcations in Case 1 (Neumann BC).

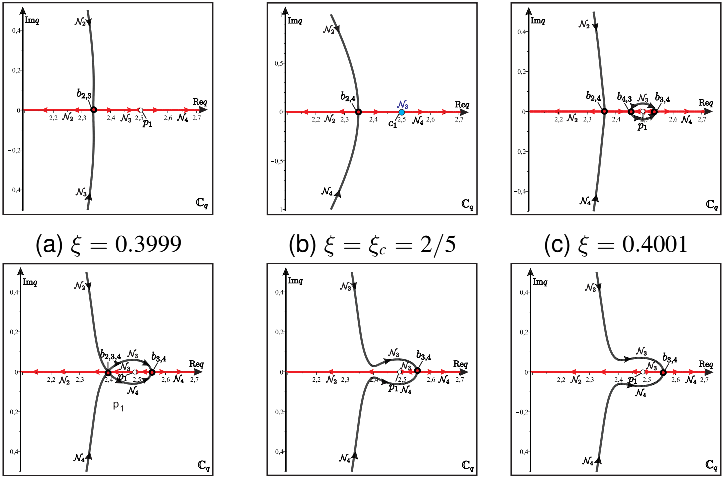
Symmetric Zero and Pole bifurcation β^0_{ZP} in Case 2.



(d) $\xi = 0.499999999$ (e) $\xi = \xi_c = 1/2$ (f) $\xi = 0.500000001$

Figure 11: CF and bifurcation in Case 2 (Neumann BC).

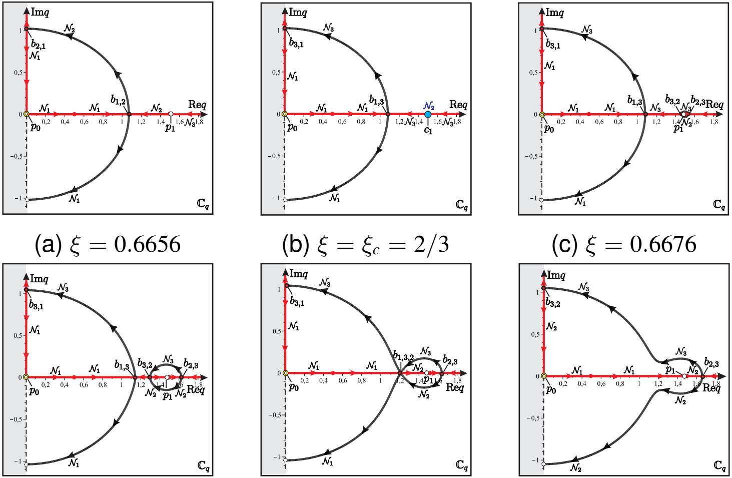
Bifurcations β_{ZP} and β_{2B} in Case 3.



(a) $\xi = 0.3999$ (b) $\xi = \xi_c = 2/5$ (c) $\xi = 0.4001$

(d) $\xi = \xi_{2b} = 0.400218...$ (e) $\xi = 0.400235$ (f) $\xi = 0.40036$

Figure 12: Bifurcations in Case 3 (Neumann BC).



(a) $\xi = 0.6656$ (b) $\xi = \xi_c = 2/3$ (c) $\xi = 0.6676$

(d) $\xi = 0.673$ (e) $\xi = \xi_{2b} = 0.67495\ldots$ (f) $\xi = 0.678$

Figure 13: Bifurcations in Case 3 (Neumann BC).

Let us analyze the SLP with one classical BC

$$-u'' = \lambda u, \quad t \in (0,1), \tag{30}$$
$$u(0) = 0, \tag{31}$$

and another two-point NC

$$u(\xi) = \gamma u(1-\xi), \tag{32}$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$.
Case $\gamma = 0$. If $\xi = 0$, then have problem (30),(31) with one BC $u(0) = 0$ only. If $0 < \xi \leq 1$, then we have the classical BVP in the interval $[0,\xi]$ with BCs $u(0) = 0$, $u(\xi) = 0$, and its eigenvalues and eigenfunctions are

$$\lambda_k = \left(\frac{\pi k}{\xi}\right)^2, \quad u_k(t) = \sin\left(\frac{\pi k t}{\xi}\right), \quad k \in \mathbb{N}. \tag{33}$$

Case $\gamma = \infty$. If $\xi = 1$, we have problem (30)–(31) with one BC $u(0) = 0$. If $0 \leq \xi < 1$ then we have the same situation as in Case $\gamma = 0$ with the BVP in the interval $[0, 1-\xi]$.
Case $\xi = \frac{1}{2}$. If $\gamma = 1$, then we have problem (30)–(31) with one BC $u(0) = 0$. If $\gamma \neq 1$, then we have BVP in the interval $[0, \frac{1}{2}]$ and the initial value problem in the interval $\left[\frac{1}{2}, 1\right]$.

There exists a nontrivial solution, if q is the root of the equation

$$\frac{\sin(\xi\pi q)}{\pi q} = \gamma\frac{\sin\left((1-\xi)\pi q\right)}{\pi q}. \quad Z_\xi(q) = \gamma P_\xi(q) = \gamma Z_{1-\xi}(q), \quad q \in \mathbb{C}_q. \tag{34}$$

Roots of the system

$$\sin(\pi q) = 0, \qquad \sin(\pi q\xi) = 0, \tag{35}$$

are CEP of SLP (30)–(32).

Lemma 2.29.

For SLP (30)–(32) Constant Eigenvalues exist only for rational parameter $\xi = m/n \in (0,\ 1)$, $m, n \in \mathbb{N}$, $\xi \neq 1/2$, values and those eigenvalues are equal to $\lambda_s = (\pi c_s)^2$, $c_s = ns$, $s \in \mathbb{N}$.

Kristina Skučaitė-Bingelė, Artūras Štikonas (2013);
"Inverstigation of the spectrum for Sturm-Liouville problems with a nonlocal boundary condition".

For SLP (30)–(32) we have meromorphic Complex CF

$$\gamma_c(q) = \frac{Z(q)}{Z_{1-\xi}(q)} = \frac{\sin(\xi\pi q)}{\sin\left((1-\xi)\pi q\right)}, \quad q \in \mathbb{C}_q. \tag{36}$$

Remark 2.31.

NC (32) we can rewrite as

$$u(1-\xi) = \tilde{\gamma} u(\xi), \qquad \tilde{\gamma} = 1/\gamma. \tag{37}$$

NC (37) we can rewrite as

$$u(\tilde{\xi}) = \tilde{\gamma} u(1-\tilde{\xi}). \tag{38}$$

So, the spectrum for SPL (30)–(32) with parameters $0 < \xi < 1/2$ and γ is the same as the spectrum for SPL (30)–(31), (37) with parameters $1/2 < \tilde{\xi} < 1$ and $\tilde{\gamma} = 1/\gamma$. Thus, it is enough to investigate problem (30)–(32) with the parameter $1/2 < \xi < 1$.

Theorem 2.33.

Spectrum for SLP (30)–(32) for $1/2 < \xi < 1$ is equivalent to Spectrum for SLP:

$$\begin{aligned} -u'' &= \lambda u, \quad t \in (0,1), \\ u(0) &= 0, \\ u(1) &= \gamma u(\xi), \end{aligned} \tag{39}$$

Artūras Štikonas (2007);
"The Sturm–Liouville problems with nonlocal boundary conditions"

Conclusions of this Chapter

1. For SLP with Dirichlet type BC (two cases SLP_1^d, SLP_2^d) (15)–(17), SLP with Neumann type BC (three cases SLP_1^n, SLP_2^n, SLP_3^n and $\mathrm{SLP}_4^n \sim \mathrm{SLP}_2^d$) (22)–(24) and SLP with symmetrical type BC (SLP^s) (30)–(32) CEs do not exist for irrational parameter ξ and exist only for rational $\xi = \frac{m}{n} \in \mathbb{Q}, 0 < m < n, \gcd(m,n) = 1$

2. CPs of the first order (and complex eigenvalues) exist for all SLP in case $\xi \in (0,1)$, for SLP_2^d in case $\xi = 0$, SLP_1^n in case $\xi = 0$. We have infinite number of such CPs of the first order. Negative CP of the first order exists for SLP_1^n in case $\xi \in (0,1)$. CPs of the second order exist for SLP_2^d, SLP_1^n, SLP_3^n (only for some $\xi \in (0,1)$).

3. For SSLP_1^d, SLP_2^d, SLP_1^n, SLP_2^n, SLP_3^n, SLP^s we obtain five types bifurcations (β_{ZP}^0, β_{2B}, β_{ZP}, β_{2B}^{-1} and β_{ZP}^{-1})

In this chapter we investigate *a discrete Sturm–Liouville Problems* (dSLP) corresponding to SLPs in Chapter 1 and Chapter 2:

$$-u'' = \lambda u, \quad t \in (0,1), \tag{40}$$

with one classical (Dirichlet or Neumann) BC:

$$u(0) = 0 \text{ or } u'(0) = 0 \tag{41}$$

and another two-point NBC:

$$u(1) = \gamma u'(\xi), \tag{42a}$$
$$u(1) = \gamma u(\xi), \tag{42b}$$

with the parameter $\gamma \in \mathbb{R}$, $0 < \xi < 1$.

Kristina Skučaitė-Bingelė, Artūras Štikonas (2011);
"Investigation of complex eigenvalues for a stationary problem with two-point nonlocal boundary condition".
Kristina Bingelė, Agnė Bankauskienė, Artūras Štikonas (2019);
"Spectrum Curves for a discrete Sturm–Liouville problem with one integral boundary condition".

We approximate SLP (40)–(42) with Diriclet BC by the following Finite-Difference Scheme (FDS) and get *a discrete Sturm–Liouville Problem* (dSLP):

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}, \tag{43}$$

$$U_0 = 0, \tag{44}$$

with NBC $(0 < m < n)$

$$U_n = \tfrac{\gamma}{2h}(U_{m+1} - U_{m-1}), \tag{45a}$$

$$U_n = \gamma U_m. \tag{45b}$$

If $\gamma = 0$, we have the classical BCs and all the $n-1$ eigenvalues for the classical FDS are positive and algebraically simple and do not depend on the parameters ξ:

$$\lambda_k(0) = \lambda^h(q_k(0)),\ U_{k,j}(0) = \sin(\pi q_k(0) t_j),\ q_k(0) = k \in \mathbb{N}^h. \tag{46}$$

$$\frac{\sin(\pi q)}{\pi q}\cdot\frac{\pi q h}{\sin(\pi q h)} = \gamma\cos(\xi\pi q), \tag{47a}$$

$$\frac{\sin(\pi q)}{\pi q}\cdot\frac{1}{1-hq} = \gamma\frac{\sin(\xi\pi q)}{\pi q}\cdot\frac{1}{1-hq}. \tag{47b}$$

Roots of this equation are EPs for dSLP (43)–(45). The bijection $\lambda = \lambda^h(q) = \frac{4}{h^2}\sin^2(\pi q h/2) = \frac{2}{h^2}\big(1-\cos(\pi z h)\big)$ allows to find corresponding eigenvalues.

We introduce functions:

$$Z^h(z) := Z(z)\cdot\frac{\pi z h}{\sin(\pi z h)},\qquad Z(z) := \frac{\sin(\pi z)}{\pi z},\ P_\xi^h(z) = P_\xi(z) := \cos(\xi\pi z) \tag{48a}$$

$$Z^h(z) := Z(z)\cdot\frac{1}{\pi z(hz-1)},\qquad Z(z) := \sin(\pi z),\ P_\xi^h(z) = P_\xi(z)\cdot\frac{1}{\pi z(hz-1)}, \tag{48b}$$

$$P_\xi(z) := \sin(\xi\pi z);$$

For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system:

$$Z^h(q) = 0,\qquad P_\xi^h(q) = 0. \tag{49}$$

Lemma 3.5.

For dSLP (43)–(45a) Constant Eigenvalues exist only for $\xi = M/N \in (0,\ 1)$, $M \in \mathbb{N}_o$, $N \in \mathbb{N}_e$, values and those eigenvalues are equal to $\lambda_s = \lambda^h(c_s)$, $c_s := (s - 1/2)N$, $s = \overline{1, K}$.

Lemma 3.6.

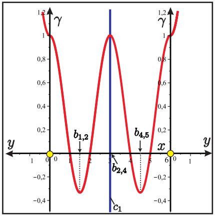
For dSLP (43)–(45b) Constant Eigenvalues exist only for $\xi = M/N \in (0,\ 1)$, $M, N \in \mathbb{N}$, $K > 1$, values and those eigenvalues are equal to $\lambda_s = \lambda^h(c_s)$, $c_s := Ns$, $s = \overline{1, K-1}$.

For dSLP (43)–(45) we have meromorphic functions (Complex CF)

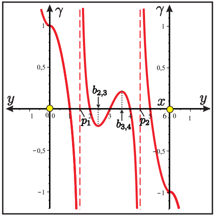
$$\gamma_c(q) := \frac{Z(q)}{P_\xi(q)} \cdot \frac{\pi q h}{\sin(\pi q h)} = \frac{\sin(\pi q)}{\pi q \cos(\xi \pi q)} \cdot \frac{\pi q h}{\sin(\pi q h)}, \tag{50a}$$

$$\gamma_c(q) := \frac{Z(q)}{P_\xi(q)} = \frac{\sin(\pi q)}{\sin(\xi \pi q)}. \tag{50b}$$

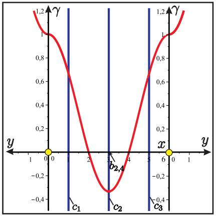
Real CF in Case 1.



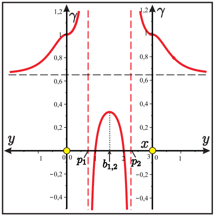
(a) $\xi = 1/6$



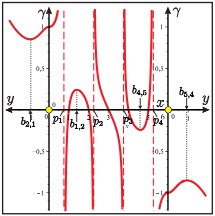
(b) $\xi = 2/6$



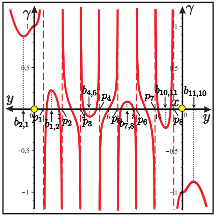
(c) $\xi = 3/6$



(d) $\xi = 2/3$

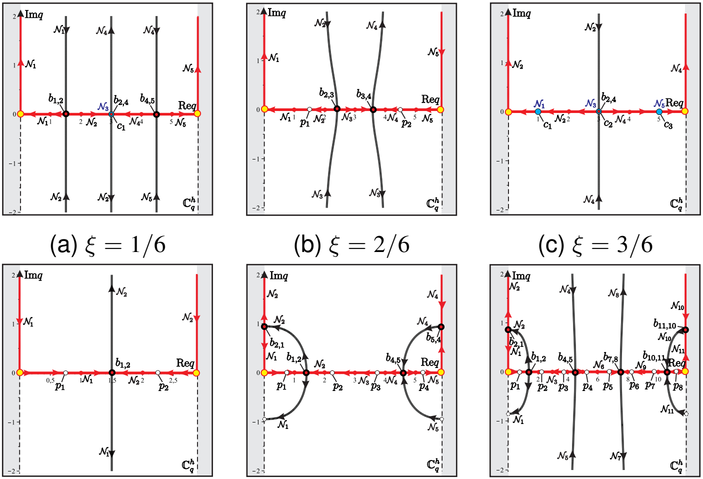


(e) $\xi = 4/6$



(f) $\xi = 8/12$

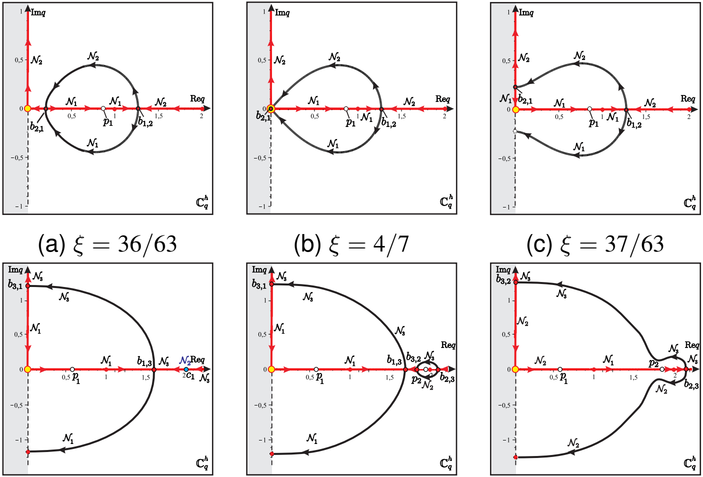
Figure 14: Real CF (dSLP Dirichlet BC) for various parameter ξ values in Case 1.



(a) $\xi = 1/6$ (b) $\xi = 2/6$ (c) $\xi = 3/6$

(d) $\xi = 2/3$ (e) $\xi = 4/6$ (f) $\xi = 8/12$

Figure 15: Spectrum Curves (dSLP Dirichlet BC) for various ξ values in Case 1.

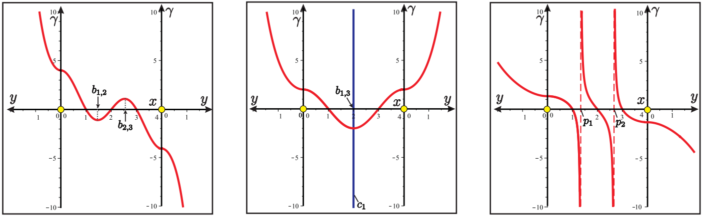


(a) $\xi = 36/63$ (b) $\xi = 4/7$ (c) $\xi = 37/63$

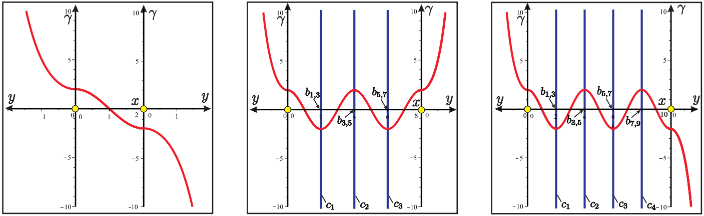
(d) $\xi = 300/400$ (e) $\xi = 301/400$ (f) $\xi = 302/400$

Figure 16: Spectrum Curves near Ramification Point $q = 0$ Case 1.

Real CF in Case 2.

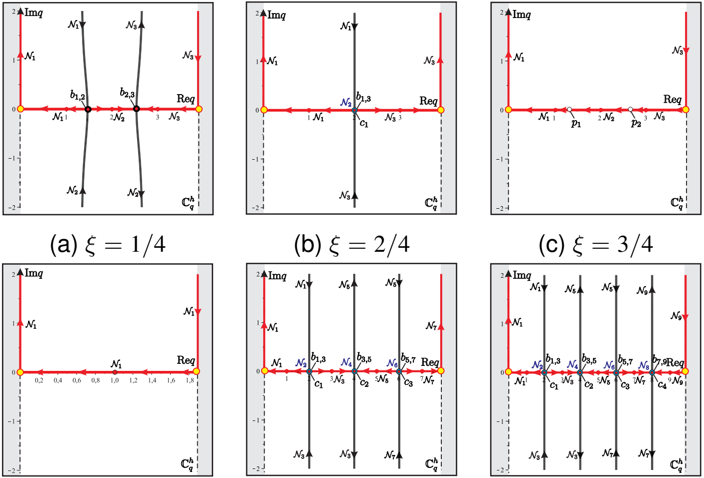


(a) $\xi = 1/4$ (b) $\xi = 2/4$ (c) $\xi = 3/4$



(d) $\xi = 1/2$ (e) $\xi = 4/8$ (f) $\xi = 5/10$

Figure 17: Real CF (dSLP Dirichlet BC) for various parameter ξ values in Case 2.



(a) $\xi = 1/4$ (b) $\xi = 2/4$ (c) $\xi = 3/4$

(d) $\xi = 1/2$ (e) $\xi = 4/8$ (f) $\xi = 5/10$

Figure 18: Spectrum Curves (dSLP Dirichlet BC) for various ξ values in Case 2.

We approximate SLP (40)–(42) with Neumann type BC by FDS and get dSLP:

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}, \tag{51}$$

$$U_0 = U_1, \tag{52}$$

with NBC $(0 < m < n)$

$$U_n = \tfrac{\gamma}{2h}(U_{m+1} - U_{m-1}), \tag{53a}$$

$$U_n = \gamma U_m. \tag{53b}$$

If $\gamma = 0$, we have the classical BCs and all the $n-1$ eigenvalues for the classical FDS are positive and algebraically simple and do not depend on the parameters ξ:

$$\lambda_k(0) = \frac{4}{h^2}\sin^2(\pi q_k(0)h/2), \quad U_{k,j}(0) = \frac{\cos\left(\pi q_k(0)(t_j - h/2)\right)}{\cos(\pi q_k(0)h/2)}, \tag{54}$$

$k = \overline{1, n-1}.$

$$-\frac{\cos\left(\pi q(1-h/2)\right)}{\cos(\pi qh/2)} = \gamma \sin\left(\pi q(\xi - h/2)\right) \cdot \frac{\sin(\pi qh/2)}{h/2}, \tag{55a}$$

$$\frac{\cos\left(\pi q(1-h/2)\right)}{\cos(\pi qh/2)} = \gamma \frac{\cos\left(\pi q(\xi - h/2)\right)}{\cos(\pi qh/2)}. \tag{55b}$$

Roots of this equation are EPs for dSLP (51)–(53).
We introduce functions:

$$Z^h(z) := \frac{\cos\left(\pi z(1-h/2)\right)}{\cos(\pi zh/2)}; \tag{56}$$

$$P^h_\xi(z) := -\sin\left(\pi z(\xi - h/2)\right) \cdot \frac{\sin(\pi zh/2)}{h/2}, \tag{57a}$$

$$P^h_\xi(z) := \frac{\cos\left(\pi z(\xi - h/2)\right)}{\cos(\pi zh/2)}. \tag{57b}$$

For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$.
CEP are roots of the system:

$$Z^h(q) = 0, \qquad P^h_\xi(q) = 0. \tag{58}$$

Lemma 3.17.

For dSLP (51)–(53a) Constant Eigenvalues do not exist.

Lemma 3.18.

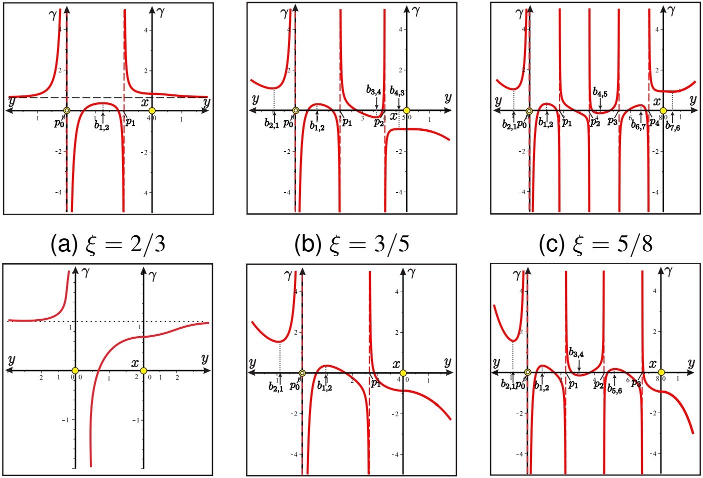
For dSLP (51)–(53b) Constant Eigenvalues exist only for $\xi = m/n \in (0,\ 1)$, $K > 1$, values and those eigenvalues are equal to $\lambda_s = \lambda^h(c_s)$, $c_s := n/K \cdot (2s-1)$, $s = \overline{1, (K-1)/2}$.

For dSLP (51)–(53) we have meromorphic functions (Complex CF)

$$\gamma_c(q) := -\frac{\cos\big(\pi q(1-h/2)\big)}{\sin\big(\pi q(\xi - h/2)\big)} \cdot \frac{h}{\sin(\pi q h)}, \tag{59a}$$

$$\gamma_c(q) := \frac{\cos\big(\pi q(1-h/2)\big)}{\cos\big(\pi q(\xi - h/2)\big)}. \tag{59b}$$

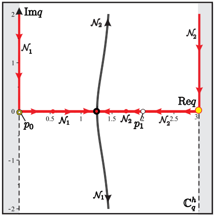
Real CF in Case 1.



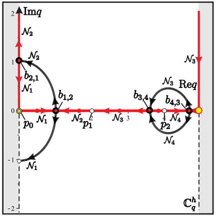
(a) $\xi = 2/3$ (b) $\xi = 3/5$ (c) $\xi = 5/8$

(d) $\xi = 1/2$ (e) $\xi = 2/4$ (f) $\xi = 4/8$

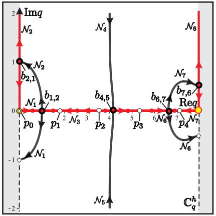
Figure 19: Real CF γ_r (dSLP Neumann BC) for various parameter ξ values in Case 1.



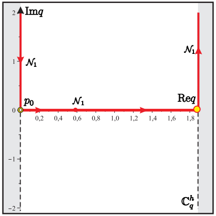
(a) $\xi = 2/3$



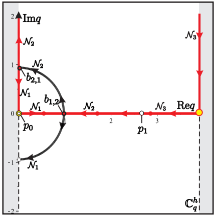
(b) $\xi = 3/5$



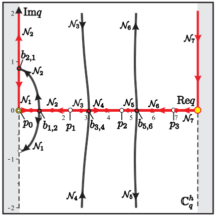
(c) $\xi = 5/8$



(d) $\xi = 1/2$



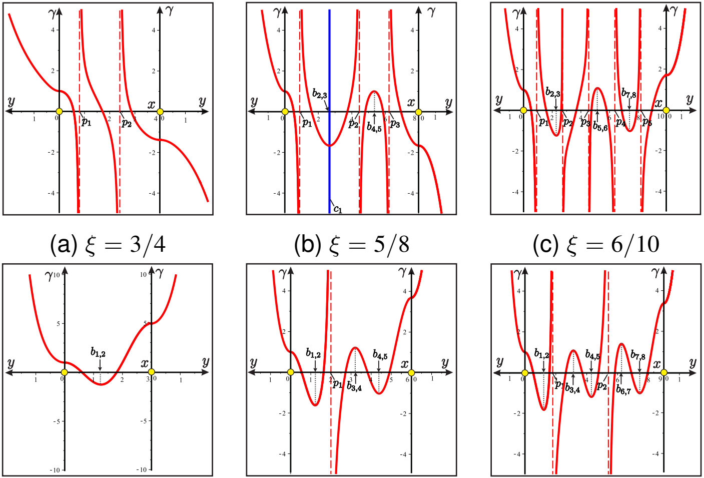
(e) $\xi = 2/4$



(f) $\xi = 4/8$

Figure 20: Spectrum Curves (dSLP Neumann BC) for various ξ values in Case 1.

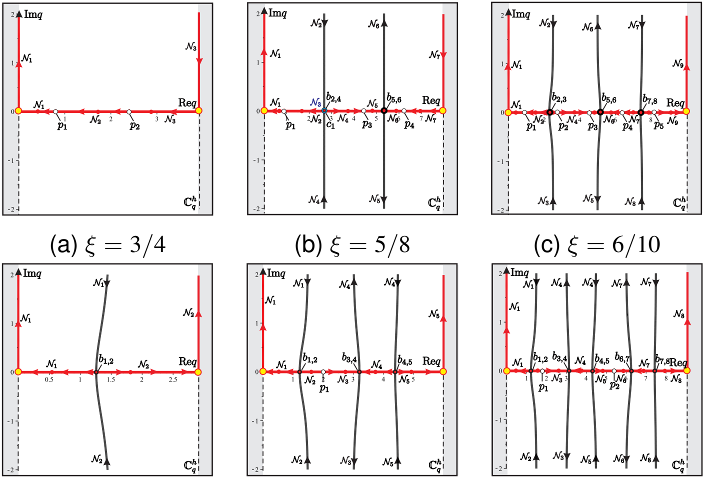
Real CF in Case 2.



(a) $\xi = 3/4$ (b) $\xi = 5/8$ (c) $\xi = 6/10$

(d) $\xi = 1/3$ (e) $\xi = 2/6$ (f) $\xi = 3/9$

Figure 21: Real CF γ_r (dSLP Neumann BC) for various parameter ξ values in Case 2.



(a) $\xi = 3/4$ (b) $\xi = 5/8$ (c) $\xi = 6/10$

(d) $\xi = 1/3$ (e) $\xi = 2/6$ (f) $\xi = 3/9$

Figure 22: Spectrum Curves (dSLP Neumann BC) for various ξ values in Case 2.

Conclusions of this Chapter

We investigate the spectrum discrete Problems: two dSLP with one classical Dirichlet BC and two-points NBCs, two dSLP with one classical Neumann type BC and two-points NBCs.

1. For dSLP with one classical Dirichlet BC CEs can exist (in Case 1 and Case 2);
2. For dSLP with one classical Neumann BC CEs can exist (in Case 1) and do not exist (in Case 2);
3. In limit case, the discrete Problems CF is the same as for differential SLPs.