Finite volume ADI schemes for hybrid dimension heat conduction models

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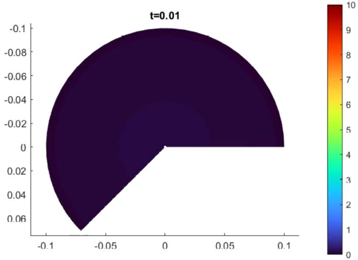
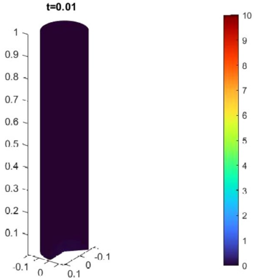
Motivational introduction (1/2)

Consider a cylinder with $r = 0.1$, height $= 1$.



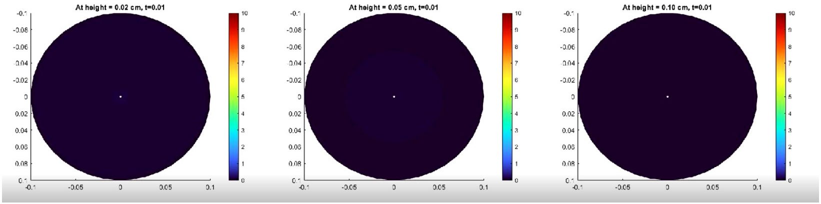


$$g_1 = 10te^{-(r/R)^2}, \quad g_2 = 0, \quad u^0 = 0.$$

Motivational introduction (2/2)

Consider its cross-sections at heights

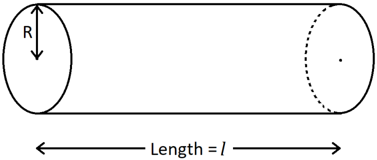
0.02, 0.05, 0.1.



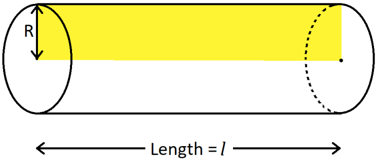
3D tube with radial symmetry

Domain

Consider a tube $\mathcal{T} \subset \mathbb{R}^3$ in cylindrical coordinates (r, ϕ, z)



and domain $\Omega = \{(r, z) \in (0, R) \times (0, l)\}$



Classical problem (1.1)

Heat equation :

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} + f(r,z,t), \ \ (r,z,t) \in \Omega_T = \Omega \times (0,T],$$

Boundary conditions :

$$u(r,0,t) = g_1(r,t), \quad u(r,l,t) = g_2(r,t), \ (r,t) \in (0,R] \times (0,T],$$

$$r\frac{\partial u}{\partial r} = 0, \quad 0 < z < l, r = 0 \text{ and } r = R, \ 0 < t \leq T,$$

Initial condition :

$$u(r,z,0) = u^0(r,z), \quad (r,z) \in \Omega.$$

Numerical mesh

The uniform spatial mesh is defined as $\bar{\Omega}_h = \bar{\omega}_r \times \bar{\omega}_z$ with

$$\bar{\omega}_r = \{r_j : \ \ r_j = jh, \ \ j = 0, \dots, J\}, \quad r_J = R,$$
$$\bar{\omega}_z = \{z_k : \ \ z_k = kH, \ \ k = 0, \dots, K\}, \quad z_K = l.$$

Here h and H are the space step sizes.
We also consider a uniform time mesh:

$$\bar{\omega}_t = \{t^n : \ \ t^n = n\tau, \qquad n = 0, \dots, N\}, \quad t^N = T,$$

here τ is the time step size.
Let U_{jk}^n be a numerical approximation at the grid point (r_j, z_k, t^n).

FVM space discretization operators

$$\partial_z U_{jk}^n := \frac{U_{jk}^n - U_{j,k-1}^n}{H}, \quad A_2^h U_{jk}^n := -\frac{1}{H}\Big(\partial_z U_{j,k+1}^n - \partial_z U_{jk}^n\Big).$$

$$\partial_r U_{jk}^n := \frac{U_{jk}^n - U_{j-1,k}^n}{h}, \quad A_1^h U_{jk}^n := -\frac{1}{\widetilde{r}_j h}\Big(r_{j+\frac{1}{2}}\partial_r U_{j+1,k}^n - r_{j-\frac{1}{2}}\partial_r U_{jk}^n\Big),$$

where

$$\widetilde{r}_0 = \frac{1}{8}h,\ \widetilde{r}_j = r_j,\ 1 \le j < J,\ \widetilde{r}_J = \frac{1}{2}\big(R - \frac{h}{4}\big),\ r_{-\frac{1}{2}} = 0,\ r_{J+\frac{1}{2}} = 0.$$

ADI for time integration

$$\frac{U_{jk}^{n+\frac{1}{2}} - U_{jk}^{n}}{\tau/2} + A_1^h U_{jk}^{n+\frac{1}{2}} + A_2^h U_{jk}^{n} = f_{jk}^{n+\frac{1}{2}},$$

$$\frac{U_{jk}^{n+1} - U_{jk}^{n+\frac{1}{2}}}{\tau/2} + A_1^h U_{jk}^{n+\frac{1}{2}} + A_2^h U_{jk}^{n+1} = f_{jk}^{n+\frac{1}{2}},$$

Consistency of the ADI scheme

Lemma

If a solution of the problem (1.1) is sufficiently smooth, then the approximation error of the ADI scheme is $O(\tau^2 + h^2 + H^2)$.

Properties of operators A_1 and A_2

Lemma

The discrete operators A_1^h and A_2^h are symmetric and positive semi-definite and positive definite operators, respectively.

First, the operator A_2^h is investigated. We get

$$(A_2^h u, v) = \sum_{k=1}^{K-1} (A_2^h u)_k v_k H = (\partial_z u, \partial_z v].$$

It follows that A_2^h is a symmetric operator. It is also well-known that the eigenvalue problem

$$A_2^h \phi_l = \lambda_l \phi_l$$

has a complete set of eigenvectors ϕ_l, $l = 1, \dots, K-1$, and all eigenvalues are positive $\lambda_l > 0$ Thus A_2^h is positive-definite.

Properties of operators A_1 and A_2

Now consider the operator A_1^h.We get

$$[A_1^h u, v]_r = \sum_{j=0}^{J} \tilde{r}_j (A_1^h u)_j v_j h = (\partial_r u, \partial_r v]_r.$$

It follows that A_1^h is a symmetric operator. From

$$[A_1^h u, u]_r = (\partial_r u, \partial_r u]_r \geq 0$$

we see that A_1^h is a positive semi-definite operator. The eigenvalue problem

$$A_1^h \psi_l = \mu_l \psi_l$$

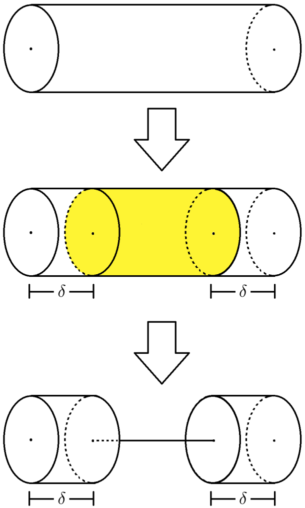
has a complete set of eigenvectors ψ_l, $l = 0, \ldots, J$, one eigenvalue $\mu_0 = 0$ and the remaining eigenvalues are positive $\mu_l > 0$.

Properties of operators A_1 and A_2

Lemma

ADI scheme is unconditionally stable.

Geometry of the hybrid dimension model



Problem for the approximate solution[1]

$$\frac{\partial U}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{\partial^2 U}{\partial z^2} + f(z,t), \ \ (r,z,t) \in (\Omega \setminus \Omega_\delta) \times (0,T],$$

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial z^2} + f(z,t), \ \ (r,z,t) \in \Omega_\delta \times (0,T],$$

Boundary conditions :

$$U(r,0,t) = g_1(r,t), \quad U(r,l,t) = g_2(r,t), \ (r,t) \in (0,R] \times (0,T],$$

$$r\frac{\partial U}{\partial r} = 0, \quad z \in (0,\delta) \cup (l-\delta, l), r = 0 \text{ and } r = R, \ 0 < t \leq T,$$

Initial condition :

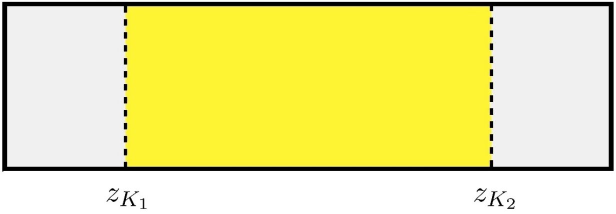
$$U(r,z,0) = u^0(r,z), \quad (r,z) \in \Omega.$$

[1] A. Amosov, G. Panasenko, Partial dimension reduction for the heat equation in a domain containing thin tubes (2018)

Mesh of the hybrid dimension model

K_1 and K_2 define the indices of truncation points: $z_{K_1} = \delta$, $z_{K_2} = l - \delta$. Then the spatial mesh ω_z is split into three parts:

$$\omega_{z1} = \{z_k : \ \ z_k = kH, \ \ k = 1, \dots, K_1 - 1\},$$
$$\omega_{z2} = \{z_k : \ \ z_k = kH, \ \ k = K_1 + 1, \dots, K_2 - 1\},$$
$$\omega_{z3} = \{z_k : \ \ z_k = kH, \ \ k = K_2 + 1, \dots, K - 1\}.$$



Averaging operator

Let S_h denote the discrete averaging operator

$$S_h(U_k^n) = \frac{2}{R^2} \sum_{j=0}^{J} \widetilde{r}_j U_{jk}^n h.$$

Conjugation conditions

$$U\Big|_{z=\delta-0} = U\Big|_{z=\delta+0}, \quad U\Big|_{z=l-\delta-0} = U\Big|_{z=l-\delta+0},$$

$$\frac{\partial S(U)}{\partial z}\Big|_{z=\delta-0} = \frac{\partial U}{\partial z}\Big|_{z=\delta+0}, \quad \frac{\partial U}{\partial z}\Big|_{z=l-\delta-0} = \frac{\partial S(U)}{\partial z}\Big|_{z=l-\delta+0}.$$

The first two conditions are classical and mean that U is continuous at the truncation points, while the remaining two conditions are nonlocal and they define the conservation of full fluxes along the separation lines.

Problem for the numerical solution

Equations of the first half-step of the ADI scheme for the hybrid dimension heat conduction problem

$$\frac{U_{jk}^{n+\frac{1}{2}} - U_{jk}^{n}}{\tau/2} + A_1^h U_{jk}^{n+\frac{1}{2}} + A_2^h U_{jk}^{n} = f_{jk}^{n+\frac{1}{2}}, \quad (r_j, z_k) \in \bar{\omega}_r \times (\omega_{z1} \cup \omega_{z3}),$$

$$\frac{U_{*k}^{n+\frac{1}{2}} - U_{*k}^{n}}{\tau/2} + A_2^h U_{*k}^{n} = f_{*k}^{n+\frac{1}{2}}, \quad z_k \in \omega_{z2},$$

$$\frac{U_{*K_1}^{n+\frac{1}{2}} - U_{*K_1}^{n}}{\tau/2} + \frac{1}{H^2}\Big(- S_h(U_{K_1-1}^{n}) + 2U_{*K_1}^{n} - U_{*,K_1+1}^{n}\Big) = f_{*K_1}^{n+\frac{1}{2}},$$

$$\frac{U_{*K_2}^{n+\frac{1}{2}} - U_{*K_2}^{n}}{\tau/2} + \frac{1}{H^2}\Big(- S_h(U_{K_2+1}^{n}) + 2U_{*K_2}^{n} - U_{*,K_2-1}^{n}\Big) = f_{*K_2}^{n+\frac{1}{2}}.$$

Existence and uniqueness of a numerical solution

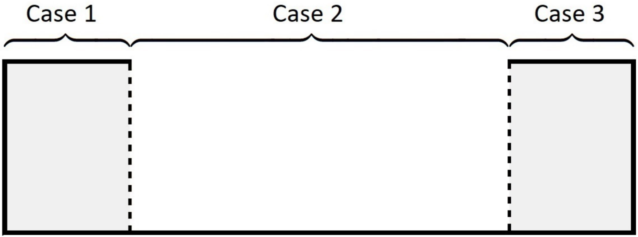
The **existence and uniqueness** of a solution to the approximate problem, as well as **validity of conjugation conditions** at truncations, was proved in the works of prof. G. Panasenko[2]by analysing the weak form of heat equation.

We have proved the existence and uniqueness of a numerical solution.

[1]A. Amosov, G. Panasenko, Partial dimension reduction for the heat equation in a domain containing thin tubes (2018)

A modified Thomas algorithm: 3 cases

The following 3 cases are considered:



In each case, the ADI equations can be written in the form

$$-a_{jk}U^{n+1}_{j,k-1} + c_{jk}U^{n+1}_{jk} - b_{jk}U^{n+1}_{j,k+1} = d_{jk},$$
$$a_{jk},\ b_{jk},\ c_{jk} \geq 0,\ \ c_{jk} \geq a_{jk} + b_{jk}.$$

Case 1

Domain ω_{z1}. The solution is presented in the following form:

$$U_{jk}^{n+1} = \alpha_{jk} U_{j,k+1}^{n+1} + \gamma_{jk}, \quad 0 \leq k < K_1,$$
$$\alpha_{j0} = 0, \ \gamma_{j0} = g_1(r_j, t^{n+1}),$$
$$\alpha_{jk} = \frac{b_{jk}}{c_{jk} - a_{jk}\alpha_{j,k-1}}, \ \gamma_{jk} = \frac{d_{jk} + a_{jk}\gamma_{j,k-1}}{c_{jk} - a_{jk}\alpha_{j,k-1}}.$$

By induction it can be proved that the estimates $0 \leq \alpha_{jk} \leq 1$ are valid.

Case 2

Domain ω_{z2}. The solution is presented in the following form:

$$U_{*k}^{n+1} = \alpha_{*k} U_{*K_1}^{n+1} + \beta_{*k} U_{*K_2}^{n+1} + \gamma_{*k}, \quad K_1 < k < K_2.$$

This factorization is done in two steps.

Case 2

First, the solution is written in the form

$$U_{*k}^{n+1} = \widetilde{\alpha}_{*k} U_{*K_1}^{n+1} + \widetilde{\beta}_{*k} U_{*,k+1}^{n+1} + \widetilde{\gamma}_{*k}, \quad K_1 < k < K_2,$$

$$\widetilde{\alpha}_{*,K_1+1} = \frac{a_{*,K_1+1}}{c_{*,K_1+1}}, \quad \widetilde{\beta}_{*,K_1+1} = \frac{b_{*,K_1+1}}{c_{*,K_1+1}}, \quad \widetilde{\gamma}_{*,K_1+1} = \frac{d_{*,K_1+1}}{c_{*,K_1+1}},$$

$$\widetilde{\alpha}_{*k} = \frac{a_{*k}}{c_{*k} - a_{*k}\widetilde{\beta}_{*,k-1}} \widetilde{\alpha}_{*,k-1}, \quad \widetilde{\beta}_{*k} = \frac{b_{*k}}{c_{*k} - a_{*k}\widetilde{\beta}_{*,k-1}},$$

$$\widetilde{\gamma}_{*k} = \frac{a_{*k}\widetilde{\gamma}_{*,k-1} + d_{*k}}{c_{*k} - a_{*k}\widetilde{\beta}_{*,k-1}}.$$

Case 2

In the second step, we compute coefficients α_{*k}, β_{*k} and γ_{*k}

$$\alpha_{*,K_2-1} = \widetilde{\alpha}_{*,K_2-1}, \quad \beta_{*,K_2-1} = \widetilde{\beta}_{*,K_2-1}, \;\; \alpha_{*k} = \widetilde{\alpha}_{*k} + \widetilde{\beta}_{*k}\alpha_{*,k+1},$$
$$\beta_{*k} = \widetilde{\beta}_{*k}\beta_{*,k+1}, \;\; \gamma_{*k} = \widetilde{\gamma}_{*k} + \widetilde{\beta}_{*k}\gamma_{*,k+1}, \;\; k = K_2 - 2, \ldots, K_1 + 1.$$

Also, the following estimates are derived

$$0 \le \alpha_{*k}, \beta_{*k} \le 1, \quad 0 \le \alpha_{*k} + \beta_{*k} \le 1.$$

Case 3

Domain ω_{z3}. For each $j = 0, \dots, J$, the solution is presented in the following form:

$$U_{jk}^{n+1} = \beta_{jk} U_{j,k-1}^{n+1} + \gamma_{jk}, \quad K_2 < k \leq K,$$
$$\beta_{jK} = 0, \ \gamma_{jK} = g_2(r_j, t^{n+1}),$$
$$\beta_{jk} = \frac{a_{jk}}{c_{jk} - b_{jk}\beta_{j,k+1}}, \ \gamma_{jk} = \frac{d_{jk} + b_{jk}\gamma_{j,k+1}}{c_{jk} - b_{jk}\beta_{j,k+1}}.$$

By induction it can be proved that the estimates $0 \leq \beta_{jk} \leq 1$ are valid.

Conjugations of one reduced dimension zone

The conjugation conditions then form the following system of linear equations:

$$\begin{cases} A_{11} U_{*K_1}^{n+1} + A_{12} U_{*K_2}^{n+1} = B_1 \\ A_{21} U_{*K_1}^{n+1} + A_{22} U_{*K_2}^{n+1} = B_2, \end{cases}$$

From the proved estimates we have that the coefficient matrix is diagonally dominant, thus unique numerical solutions exist.

Each additional reduced dimension zone gives 2 extra equations, but the coefficient matrix remains tridiagonal.

Hybrid dimension operators $\mathcal{A}_1^h$, $\mathcal{A}_2^h$

Let us define two operators

$$\mathcal{A}_1^h U = \begin{cases} A_1^h U_{jk}, & (r_j, z_k) \in \bar{\omega}_r \times (\omega_{z1} \cup \omega_{z3}), \\ 0, & z_k \in \bar{\omega}_{z2}, \end{cases}$$

$$\mathcal{A}_2^h U = \begin{cases} A_2^h U_{jk}, & (r_j, z_k) \in \bar{\omega}_r \times (\omega_{z1} \cup \omega_{z3}), \\ A_2^h U_{*k}, & z_k \in \omega_{z2}, \\ \frac{1}{H^2}\Big(- S_h(U_{K_1-1}) + 2U_{*K_1} - U_{*,K_1+1}\Big), & k = K_1, \\ \frac{1}{H^2}\Big(- S_h(U_{K_2+1}) + 2U_{*K_2} - U_{*,K_2-1}\Big), & k = K_2. \end{cases}$$

Properties of $\mathcal{A}_1^h$, $\mathcal{A}_2^h$

Lemma

The discrete operators $\mathcal{A}_1^h$ and $\mathcal{A}_2^h$ are symmetric and positive semi-definite and positive definite operators, respectively.

First, the operator $\mathcal{A}_1^h$ is investigated. Applying the summation by part formula, we get

$$(\mathcal{A}_1^h U, V) = \sum_{j=1}^{J} r_{j-\frac{1}{2}} \Big(\sum_{k=1}^{K_1-1} \partial_r U_{jk} \partial_r V_{jk} H + \sum_{k=K_2+1}^{K-1} \partial_r U_{jk} \partial_r V_{jk} H \Big) h$$
$$= (U, \mathcal{A}_1^h V), \; (\mathcal{A}_1^h U, U) \geq 0.$$

It follows from the obtained estimates that $\mathcal{A}_1^h$ is a symmetric and positive semi-definite operator.

Properties of $\mathcal{A}_1^h$, $\mathcal{A}_2^h$

Next, the operator $\mathcal{A}_2^h$ is investigated. We prove that

$$(\mathcal{A}_2^h U, V) = (U, \mathcal{A}_2^h V),$$
$$(\mathcal{A}_2^h U, U) \geq 0.$$

Therefore, $\mathcal{A}_2^h$ is a symmetric and positive semi-definite operator. From the ellipticity condition it follows that $\mathcal{A}_2^h$ is positive definite.

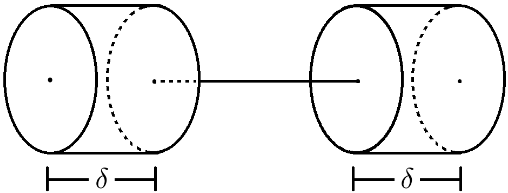
Stability estimate

Lemma

If U^n is the solution of ADI scheme, when $f^n \equiv 0$ and $g_1^n = g_2^n \equiv 0$, then the following stability estimate is valid

$$\|(I + \frac{\tau}{2}\mathcal{A}_2^h)U^n\| \leq \|(I + \frac{\tau}{2}\mathcal{A}_2^h)U^0\|.$$

Test problem (2 nodes & 1 edge)



Parameters: $l = 1,\ R = 0.1,\ T = 1,\ J = 100,\ K = 400;$

Functions: $u^0(r,z) = 0,\ g_1(r,t) = (1+3t)e^{-(r/R)^2},$

$$g_2(r,t) = te^{-(2r/R)^2},\ f(r,z,t) = 0.$$

Test problem (2 nodes & 1 edge)

$$e(\tau) = \max_{(r_j, z_k) \in \Omega_h} \left| U_{jk}^N - U(r_j, z_k, T) \right|, \quad \rho(\tau) = \log_2 \left(e(2\tau) / e(\tau) \right),$$

Table: Errors $e(\tau)$ and experimental convergence rates $\rho(\tau)$ for the discrete solution of ADI scheme for a sequence of time steps τ.

τ	$e(\tau)$	$\rho(\tau)$
0.0025	5.215e-3	1.631
0.00125	1.334e-3	1.958
0.000625	3.343e-4	2.006
0.0003125	8.194e-5	2.028

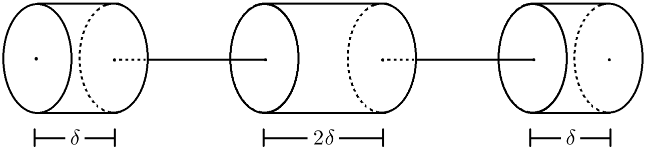
Accuracy of the reduced dimension model (2N & 1E)

$$e(\delta) = \max_{(r_j, z_k) \in \Omega_h h} \left| U_{jk}^N - U_{jk}^N(\delta) \right|$$

	$\delta = 0.05$	$\delta = 0.1$	$\delta = 0.15$	$\delta = 0.2$	$\delta = 0.25$
$e(\delta)$	0.2471	0.0377	0.0056	0.00083	0.00013

CPU time for computing the full model solution is 11.4 seconds, while for the reduced dimension model and $\delta = 0.25$ the time is reduced to 5.9 seconds, for $\delta = 0.1$ the CPU time is reduced to 2.5 seconds. $(J = 100,\ K = 800)$

Speedup of the reduced dimension model (3N & 2E)



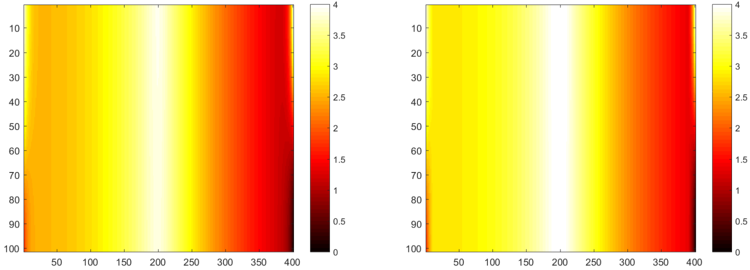
$J = 100,\ K = 1600,$

$$f(r,z,t) = 50t \exp\left(-\left(\frac{r}{R}\right)^2\right) \exp\left(-\left(\frac{z-1}{0.05}\right)^2\right) \text{ around centre.}$$

	$\delta = \delta^*$	$\delta = 0.25$	$\delta = 0.2$	$\delta = 0.15$	$\delta = 0.1$
CPU time (δ)	24.9	17.0	14.6	12.2	9.8

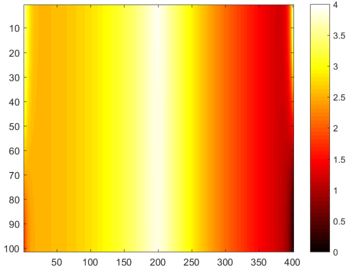
Visual comparison: 2 reduced dimension zones

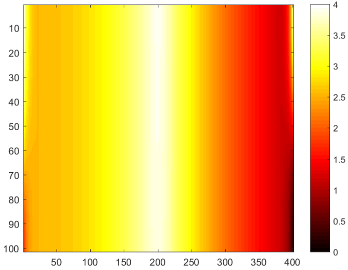
Full and reduced dimension models with $\delta = 0.05$. Which is which?



Visual comparison: 2 reduced dimension zones

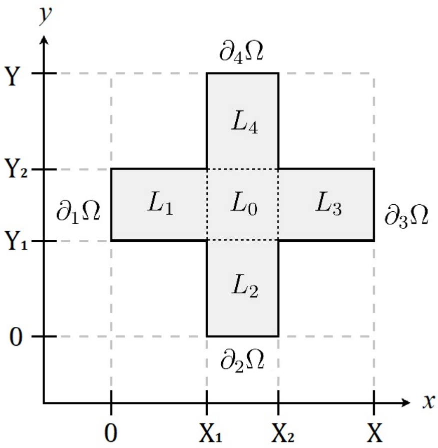
Full and reduced dimension models with $\delta = 0.1$ Which is which?





2D cross-shaped domain

Domain Ω



$$\partial_D\Omega = \partial_1\Omega \cup \partial_2\Omega \cup \partial_3\Omega \cup \partial_4\Omega$$

$$\partial_N\Omega = \partial\Omega \setminus \partial_D\Omega$$

PDE problem (2.1)

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + f(x,y,t), & (x,y,t) \in \Omega_T = \Omega \times (0,T], \\
u(0,y,t) = g_1(y,t), & (y,t) \in [Y_1, Y_2] \times (0,T], \\
u(x,0,t) = g_2(x,t), & (x,t) \in [X_1, X_2] \times (0,T], \\
u(X,y,t) = g_3(y,t), & (y,t) \in [Y_1, Y_2] \times (0,T], \\
u(x,Y,t) = g_4(x,t), & (x,t) \in [X_1, X_2] \times (0,T], \\
\dfrac{\partial u}{\partial \mathbf{n}} = 0, & (x,y,t) \in \partial_N \Omega \times (0,T], \\
u(x,y,0) = u_0(x,y), & (x,y) \in \Omega,
\end{cases}
$$

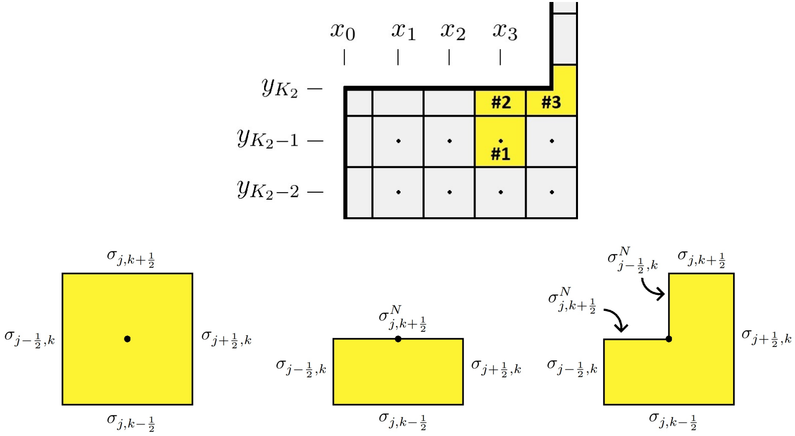
here f is a source function and $\mathbf{n}$ denotes the outer normal to $\partial_N \Omega$.

Finite Volume Method

$$u_{j,k}(t) := \frac{1}{V_{j,k}} \iint\limits_{\mathcal{K}_{j,k}} u(x,y,t)\mathrm{d}x\mathrm{d}y, \; f_{j,k}(t) := \frac{1}{V_{j,k}} \iint\limits_{\mathcal{K}_{j,k}} f(x,y,t)\mathrm{d}x\mathrm{d}y.$$

Here $\mathcal{K}_{j,k}$ is a control volume, $V_{j,k}$ is its volume, $j = 0, \ldots, J, \; k = 0, \ldots, K$.

FVM (cell-centred)



Spatial discretization of the heat equation

$$\frac{du_{j,k}}{dt} = -A_1 u_{j,k} - A_2 u_{j,k} + f_{j,k},$$

$$A_1 u_{j,k} = \frac{1}{V_{j,k}} \left(-s_{j+\frac{1}{2},k} \frac{u_{j+1,k} - u_{j,k}}{h} + s_{j-\frac{1}{2},k} \frac{u_{j,k} - u_{j-1,k}}{h} \right),$$

$$A_2 u_{j,k} = \frac{1}{V_{j,k}} \left(-s_{j,k+\frac{1}{2}} \frac{u_{j,k+1} - u_{j,k}}{H} + s_{j,k-\frac{1}{2}} \frac{u_{j,k} - u_{j,k-1}}{H} \right).$$

Here h and H are space step sizes, $s_{\alpha,\beta} = m(\sigma_{\alpha,\beta})$,

ADI scheme for numerical time integration

$$\frac{U_{j,k}^{n+\frac{1}{2}} - U_{j,k}^{n}}{\tau/2} + A_1 U_{j,k}^{n+\frac{1}{2}} + A_2 U_{j,k}^{n} = f_{j,k}^{n+\frac{1}{2}},$$

$$\frac{U_{j,k}^{n+1} - U_{j,k}^{n+\frac{1}{2}}}{\tau/2} + A_1 U_{j,k}^{n+\frac{1}{2}} + A_2 U_{j,k}^{n+1} = f_{j,k}^{n+\frac{1}{2}}.$$

Here τ is time step size, $n = 1, \ldots, N$.

Consistency

Lemma

If a solution of (1) is sufficiently smooth, then the approximation error of ADI scheme is $O(\tau^2 + h^2 + H^2)$.

Properties of operators A_1 and A_2

Lemma

The discrete operators A_1 and A_2 are symmetric and non-negative definite.

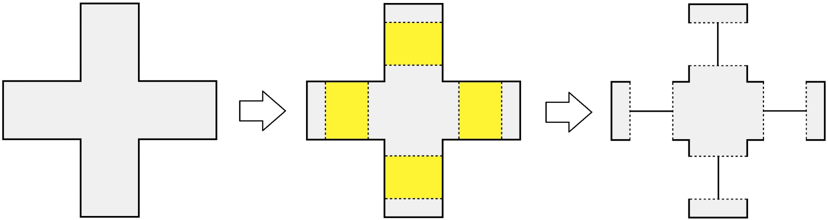
Stability

Lemma

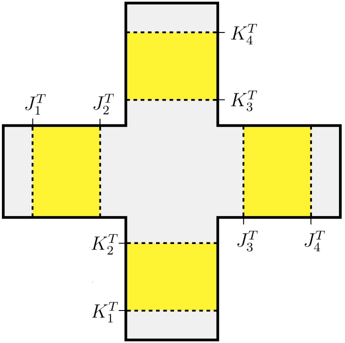
If U^n is the solution of ADI scheme, when $f^n \equiv 0$ and $g_i^n \equiv 0$, $i = 1, \ldots, 4$, then the following stability estimate is valid

$$\|(I + \frac{\tau}{2}A_2)U^n\| \leq \|(I + \frac{\tau}{2}A_2)U^0\|.$$

Geometry of the hybrid dimension model



Geometry of the hybrid dimension model



For $(x, y, t) \in (L_2 \cup L_4) \times (0, T]$:

$$u_0(x, y) = \widetilde{u}_0(y), \; f(x, y, t) = \widetilde{f}(y, t)$$

For $(x, y, t) \in (L_1 \cup L_3) \times (0, T]$:

$$u_0(x, y) = \widetilde{u}_0(x), \; f(x, y, t) = \widetilde{f}(x, t).$$

The approximate problem[3]

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + f(x,y,t), & (x,y,t) \in (\Omega \setminus \Omega^\delta) \times (0,T], \\ \dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2} + \widetilde{f}(x,t), & (x,y,t) \in (L_1^\delta \cup L_3^\delta) \times (0,T], \\ \dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial y^2} + \widetilde{f}(y,t), & (x,y,t) \in (L_2^\delta \cup L_4^\delta) \times (0,T], \\ U(0,y,t) = g_1(y,t), & (y,t) \in [Y_1, Y_2] \times (0,T], \\ U(x,0,t) = g_2(x,t), & (x,t) \in [X_1, X_2] \times (0,T], \\ U(X,y,t) = g_3(y,t), & (y,t) \in [Y_1, Y_2] \times (0,T], \\ U(x,Y,t) = g_4(x,t), & (x,t) \in [X_1, X_2] \times (0,T], \\ \dfrac{\partial U}{\partial \mathbf{n}} = 0, & (x,y,t) \in \partial_N \Omega \times (0,T], \\ U(x,y,0) = u_0(x,y), & (x,y) \in \Omega. \end{cases}$$

[1] A. Amosov, G. Panasenko, Partial dimension reduction for the heat equation in a domain containing thin tubes (2018)

Truncation of dimension

The following averaging operators are used:

$$S_x(U) = \frac{1}{X_2 - X_1} \int_{X_1}^{X_2} U(x,y,t)dx, \;\; S_y(U) = \frac{1}{Y_2 - Y_1} \int_{Y_1}^{Y_2} U(x,y,t)dy.$$

We assume continuity and conservation of full fluxes, thus conjugation conditions of the following form are used:

$$U|_{x=\delta-0} = U|_{x=\delta+0}\,, \quad \left.\frac{\partial S_y(U)}{\partial x}\right|_{x=\delta-0} = \left.\frac{\partial U}{\partial x}\right|_{x=\delta+0}.$$

Here the dimension truncation takes place at $x = \delta$.

Discrete averaging operators

$$S_x^h(U_k^n) = \frac{1}{X_2 - X_1} \sum_{j=J_1}^{J_2} U_{j,k}^n s_{j,k+\frac{1}{2}},$$

$$S_y^H(U_j^n) = \frac{1}{Y_2 - Y_1} \sum_{k=K_1}^{K_2} U_{j,k}^n s_{j+\frac{1}{2},k}.$$

Existence and uniqueness of a numerical solution

The **existence and uniqueness** of a solution to the approximate problem, as well as **validity of conjugation conditions** at truncations, was proved in the works of prof. G. Panasenko[4]by analysing the weak form of heat equation.

We have proved the existence and uniqueness of a numerical solution.

[1]A. Amosov, G. Panasenko, Partial dimension reduction for the heat equation in a domain containing thin tubes (2018)

Towards tridiagonal matrices

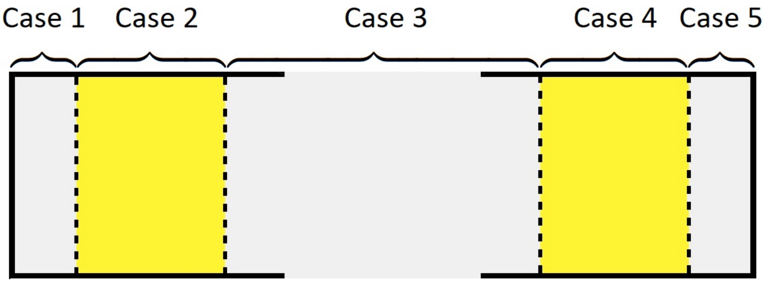
Discrete equations that are in action can be represented in the following form:

$$- a_{jk} U_{j,k-1}^{n+1} + b_{jk} U_{j,k}^{n+1} - c_{jk} U_{j,k+1}^{n+1} = d_{jk},$$

$$a_{jk}, b_{jk}, c_{jk} \geq 0, \;\; b_{jk} \geq a_{jk} + c_{jk}.$$

A modified Thomas algorithm: 5 cases

The following 5 cases are considered for $Y_1 \leq y \leq Y_2$.



A modified Thomas algorithm

The 4 conjugation conditions result in the following problem

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} U^{n+\frac{1}{2}}_{J_1^T,*} \\ U^{n+\frac{1}{2}}_{J_2^T,*} \\ U^{n+\frac{1}{2}}_{J_3^T,*} \\ U^{n+\frac{1}{2}}_{J_4^T,*} \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix},$$

From the estimates of cases 1-5, the coefficient matrix is shown to be diagonally dominant.

A modified Thomas algorithm

Theorem

A unique numerical ADI solution to the hybrid dimension heat equation model exists and can be computed using the efficient factorization algorithm.

Analysed cases 1-5, combined with conjugation conditions, prove the theorem and define the constructive algorithm to implement the method computationally.

Consistency and stability

The operators $\mathcal{A}_1^h$ and $\mathcal{A}_2^h$ are redefined by

$$\mathcal{A}_1^h U = \begin{cases} A_1^h U_{jk}, \quad (x_j, y_k) \in \Omega_h \setminus \omega_R, \\ A_1^h U_{j*}, \quad j \in (J_1^T, J_2^T) \cup (J_3^T, J_4^T), \\ 0, \quad k \in [K_1^T, K_2^T] \cup [K_3^T, K_4^T] \\ \frac{1}{h^2}\Big(- S_y^H(U_{j-1}) + 2U_{j*} - U_{j+1,*}\Big), \quad j \in \{J_1^T, J_3^T\}, \\ \frac{1}{h^2}\Big(- S_y^H(U_{j+1}) + 2U_{j*} - U_{j-1,*}\Big), \quad j \in \{J_2^T, J_4^T\}, \end{cases}$$

$$\mathcal{A}_2^h U = \begin{cases} A_2^h U_{jk}, \quad (x_j, y_k) \in \Omega_h \setminus \omega_R, \\ A_2^h U_{*k}, \quad k \in (K_1^T, K_2^T) \cup (K_3^T, K_4^T), \\ 0, \quad j \in [J_1^T, J_2^T] \cup [J_3^T, J_4^T] \\ \frac{1}{H^2}\Big(- S_x^h(U_{k-1}) + 2U_{*k} - U_{*,k+1}\Big), \quad k \in \{K_1^T, K_3^T\}, \\ \frac{1}{H^2}\Big(- S_x^h(U_{k+1}) + 2U_{*k} - U_{*,k-1}\Big), \quad k \in \{K_2^T, K_4^T\}. \end{cases}$$

Consistency and stability

Lemma

The discrete operators $\mathcal{A}_1^h$ and $\mathcal{A}_2^h$ are symmetric and positive semi-definite operators.

Consistency remains of the same order as in the classical model.

The same stability estimate is achieved as in the classical model.

$$\|(I + \frac{\tau}{2}\mathcal{A}_2^h)U^n\| \leq \|(I + \frac{\tau}{2}\mathcal{A}_2^h)U^0\|.$$

Time integration error definition

The error $e(\tau)$ and experimental convergence rate $\rho(\tau)$ at time $t = T$ are defined in the following maximum norm:

$$e(\tau) = \max_{(x_j, y_k) \in \omega} \left| U_{j,k}^N - U(x_j, y_k, T) \right|, \ \rho(\tau) = \log_2 \left(\frac{e(2\tau)}{e(\tau)} \right).$$

First test problem

Constants: $J = K = 600$, $T = 1$, $X = 1$, $Y = 1$, $X_1 = Y_1 = 1/3$, $X_2 = Y_2 = 2/3$.
Functions $u_0 = 0$, $f(x, y, t) = 0$, $g_1(y, t) = (1 + 4t)e^{-y^2}$, $g_2(x, t) = 7te^{-4x^2}$, $g_3(y, t) = 3 - 50(y - Y_1)(y - Y_2)$, $g_4(x, t) = e^t e^{-20(x-X_1)(x-X_2)}$.
The benchmark solution was computed using $\tau = 2.5 \cdot 10^{-5}$.

τ	$e(\tau)$	$\rho(\tau)$
0.0008	$6.9300 \cdot 10^{-4}$	6.2786
0.0004	$1.2685 \cdot 10^{-4}$	2.4497
0.0002	$3.1460 \cdot 10^{-5}$	2.0115
0.0001	$7.4977 \cdot 10^{-6}$	2.0690

The accuracy of reduced dimension model

Here the difference between solution to the full model and solution to the reduced dimension model is tracked, thus the following error definition is used

$$e(\delta) = \max_{(x_j, y_k) \in \omega} \left| U_{j,k}^N - U_{j,k}^N(\delta) \right|.$$

The accuracy of reduced dimension model

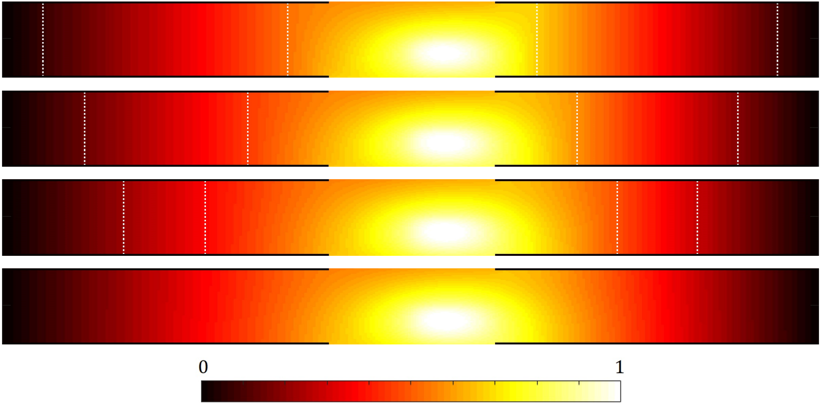
Functions of the first test problem were used for the reduced dimension model with $J = K = 300$, $X = Y = 1$ $X_1 = Y_1 = 0.45$, $X_2 = Y_2 = 0.55$ and $\tau = 0.001$.

δ	full model	0.175	0.125	0.075	0.025
$e(\delta)$	0	2.242e-3	9.194e-3	4.682e-2	2.082e-1
CPU time (s)	9.4	8.2	6.7	5.3	3.9

Considering numerous simulations, for most cases by setting δ equal to the diameter of the rod we make the dimension-reduction error $e(\delta)$ equal to approximately 1% of the cell's value at which it is found.

A visual comparison (δ : 0.05, 0.1, 0.15; *full*)

With the only nonzero function f, which equals $f(x, y, t) = 100e^t \cos(4\pi x)\cos(4\pi y)$ near centre:



Supplementary chapter: a 4th order PDE

Introduction

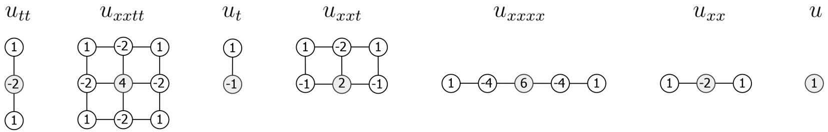
One of the problems that were encountered in the project "Multiscale Mathematical and Computer Modeling for Flows in Networks: Application to Treatment of Cardiovascular Diseases"[5] is analysed from the perspective of numerical mathematics.

We are interested to solve a 4th order PDE with constant coefficients c_i for the averaged velocity u

$$c_1 u_{tt} + c_2 u_{xxtt} + c_3 u_t + c_4 u_{xxt} + c_5 u_{xxxx} + c_6 u_{xx} + c_7 u = 0.$$

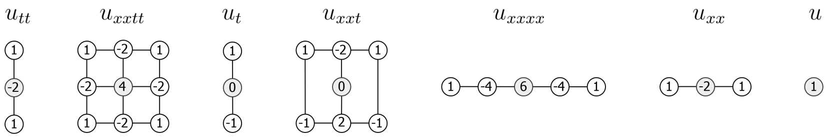
[5] Project No 09.3.3-LMT-K-712-17-0003, www.hemodynamics.mif.vu.lt

Scheme 1: central and forward differences



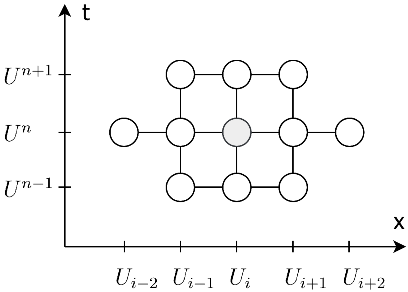
The expected accuracy of this scheme is $O(h^2 + \tau)$, here h is space step size, τ is time step size.

Scheme 2: central differences



The expected accuracy of this scheme is $O(h^2 + \tau^2)$, here h is space step size, τ is time step size.

Computational molecule of both schemes



The scheme is implemented implicitly and solved with the tridiagonal Thomas algorithm.

Accuracy of schemes

The classical method of Taylor expansions was used to show that:

The error of scheme 1 (central and forward differences) is $O(h^2 + \tau)$.

The error of scheme 2 (central differences) is $O(h^2 + \tau^2)$.

Fourier stability analysis for scheme 1

$$\epsilon_m(x,t) = E_m(t)e^{ik_m x}, \;\; G = \frac{E_m(t+\tau)}{E_m(t)},$$

$$m_1(G-2+G^{-1}) + m_2(G-2+G^{-1})(e^{-i\theta}-2+e^{i\theta}) + m_3(G-1) \\ +m_4(G-1)(e^{-i\theta}-2+e^{i\theta}) + m_5(e^{-2i\theta}-4e^{-i\theta}+6-4e^{i\theta}+e^{2i\theta}) \\ +m_6(e^{-i\theta}-2+e^{i\theta}) + m_7 = 0,$$

$$\dots$$

$$G^2A + GB + C = 0$$

Stability restriction for scheme 1

With scheme 1, for practical problems of our interest, the stability of numerical scheme is easy to satisfy with

$$\tau \leq \frac{h^2}{8},$$

this estimate was derived for parameters of a small elastic human arteriole.

Stability restriction for scheme 2

Repeating the same analysis for scheme 2, for the problems of our interest the stability is almost impossible to satisfy. E.g., for parameters of a small human arteriole stability is possible with

$$h = 0.1, \ \tau = 10^{-21}.$$

Why?

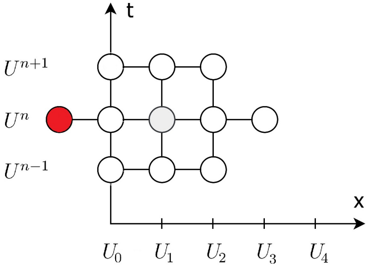
Behind the difficulties of stability for scheme 2

Recall that the Richardson scheme (only central differences) for parabolic equations is unconditionally unstable:

$$u_t = u_{xx} \xrightarrow{Richardson\ \ scheme} \frac{U_i^{n+1} - U_i^{n-1}}{2\tau} = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}$$

Hyperbolic differential terms of our PDE regularize this instability to some extent, however, their coefficients are of much lower order in magnitude, compared to parabolic terms.

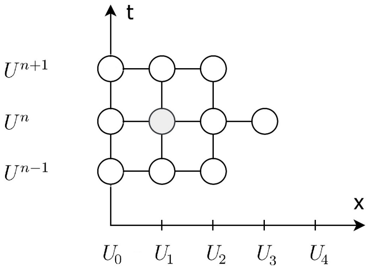
Neumann BC U_x requires to deal with *ghost* points



Ghost points can be eliminated using, e.g., the following approximation of derivative

$$U_x(x_i, t_n) = \frac{U_1^n - U_{-1}^n}{2h}.$$

BC U_{xx} does not use *ghost* points



$$U_{xxxx}(x_1, t_n) = \frac{U_{xx}(x_0, t_n) - 2U_{xx}(x_1, t_n) + U_{xx}(x_2, t_n)}{h^2}$$

$$= \frac{U_{xx}(x_0, t_n)}{h^2} + \frac{-2U(x_0, t_n) + 5U(x_1, t_n) - 4U(x_2, t_n) + U(x_3, t_n)}{h^4}.$$

Definitions of error and experimental convergence rates

In test problems, the error $e(h, \tau)$ and experimental convergence rates $\rho_\tau(\tau)$, $\rho_h(h)$ at time $t = T$ are defined in the following maximum norm:

$$e(h, \tau) = \max_i \left| U_i^{N_\tau} - U(x_i, T) \right| ,$$

$$\rho_h(h) = \log_2 \left(\frac{e(2h, \tau)}{e_h(h, \tau)} \right) , \rho_\tau(\tau) = \log_2 \left(\frac{e(h, 2\tau)}{e(h, \tau)} \right) .$$

Here $U(x_i, T)$ is a benchmark solution and N_τ is the index of some time T.

First test problem: theoretical and experimental convergence results agree well

$U(x,0) = U_t(x,0) = 0,\ U(0,t) = U(L,t) = 1 - \cos t,$
$U_{xx}(0,t) = U_{xx}(L,t) = \sin t.$

h	$e(h)$	$\rho_h(h)$
0.1	$2.3394 \cdot 10^{-5}$	1.9983
0.05	$5.8555 \cdot 10^{-6}$	2.0007
0.025	$1.4632 \cdot 10^{-6}$	2.0047
0.0125	$3.6460 \cdot 10^{-7}$	2.0188

Table: Computational results of errors in space at $T = 0.1$.

These calculations were performed using $\tau = 2^{-2} \cdot 10^{-6}$. Here the benchmark solution was calculated with $h = 2^{-6} \cdot 10^{-1}$.

First test problem: theoretical and experimental convergence results agree well

τ	$e(\tau)$	$\rho_\tau(\tau)$
0.0001	$2.3969 \cdot 10^{-7}$	1.0109
0.00005	$1.1894 \cdot 10^{-7}$	1.0228
0.000025	$5.8537 \cdot 10^{-8}$	1.0461
0.0000125	$2.8327 \cdot 10^{-8}$	1.0406

Table: Computational results of errors in time at $T = 0.1$.

Here we have used $h = 2^{-2} \cdot 10^{-1}$ and the benchmark solution was calculated with $\tau = 2^{-7} \cdot 10^{-4}$.

Second test problem: complex-exponential test

Assume that $\tilde{U} = e^{i(kx+t)} = \cos(kx+t) + i\sin(kx+t)$. The parameter k can be found by substituting $\tilde{U}$ into equation:

$$c_1\tilde{U}_{tt} + c_2\tilde{U}_{xxtt} + c_3\tilde{U}_t + c_4\tilde{U}_{xxt} + c_5\tilde{U}_{xxxx} + c_6\tilde{U}_{xx} + c_7\tilde{U} = 0.$$

By calculating derivatives and grouping terms, the following quartic equation is obtained

$$k^4c_5 + k^2(c_2 - ic_4 - c_6) + (-c_1 + ic_3 + c_7) = 0,$$

Thus we get k_1, k_2, k_3, k_4.

Case 1: Dirichlet and Neumann boundary conditions

$$\tilde{U} = e^{i(kx+t)}$$

$$\Longrightarrow$$

$$\tilde{U}(x,0) = e^{ikx},\ \tilde{U}_t(x,0) = ie^{ikx},$$
$$\tilde{U}(0,t) = e^{it},\ \tilde{U}(L,t) = e^{i(kL+t)},$$
$$\tilde{U}_x(0,t) = ike^{it},\ \tilde{U}_x(L,t) = ike^{i(kL+t)}.$$

Denote the solution acquired from these conditions with k_1 by V_1.

Case 2: Dirichlet and U_{xx} boundary conditions

$$\tilde{U} = e^{i(kx+t)}$$

$$\Longrightarrow$$

$$\begin{gathered}\tilde{U}(x,0) = e^{ikx},\ \tilde{U}_t(x,0) = ie^{ikx},\\ \tilde{U}(0,t) = e^{it},\ \tilde{U}(L,t) = e^{i(kL+t)},\\ \tilde{U}_{xx}(0,t) = -k^2 e^{it},\ \tilde{U}_{xx}(L,t) = -k^2 e^{i(kL+t)}.\end{gathered}$$

Denote the solution acquired from these conditions with k_1 by V_1^*.

Errors of V_1 and V_1^*

T	e of $Re(V_1)$	e of $Re(V_1^*)$	e of $Im(V_1)$	e of $Im(V_1^*)$
1	$6.4903 \cdot 10^{-7}$	$6.3309 \cdot 10^{-6}$	$1.8334 \cdot 10^{-6}$	$5.1999 \cdot 10^{-6}$
2	$1.1960 \cdot 10^{-6}$	$2.6706 \cdot 10^{-6}$	$1.5366 \cdot 10^{-6}$	$8.0973 \cdot 10^{-6}$
3	$1.9371 \cdot 10^{-6}$	$7.2964 \cdot 10^{-6}$	$1.3013 \cdot 10^{-7}$	$4.0185 \cdot 10^{-6}$
4	$9.0119 \cdot 10^{-7}$	$6.9911 \cdot 10^{-6}$	$1.7234 \cdot 10^{-6}$	$4.3136 \cdot 10^{-6}$
5	$9.6898 \cdot 10^{-7}$	$2.9784 \cdot 10^{-6}$	$1.6895 \cdot 10^{-6}$	$8.1424 \cdot 10^{-6}$
10	$1.3468 \cdot 10^{-6}$	$7.8812 \cdot 10^{-6}$	$1.4050 \cdot 10^{-6}$	$2.7121 \cdot 10^{-6}$

Table: Errors e of real and imaginary parts of numerical solutions V_1 and V_1^* at various times T.

Summary of the main part

- Existence and uniqueness of numerical solutions were proved for two non-classical heat conduction models.
- Constructive algorithms to implement computations were detailed.
- The convergence of ADI schemes was unaffected by dimension reduction.
- The effectiveness of strategy to fasten computations by dimension reduction was confirmed.
- The ADI method is well compatible with the Method of Asymptotic Partial Domain Decomposition for a variety of heat conduction problems.

Summary of the supplementary part

For the 4th order PDE with constant coefficients:

- The scheme constructed with central differences has higher accuracy in time compared to the scheme constructed with central and forward differences.
- However, in practical computations the latter has better performance due to lesser restrictions on step sizes.
- Choosing boundary conditions of type U_{xx} instead of U_x gives considerably smaller errors.

Publications

1. R. Čiegis, G. Panasenko, K. Pileckas, V. Šumskas, *ADI scheme for partially dimension reduced heat conduction models*, Comput. Math. with Appl., **80** (5):1275-1286, 2020.

2. V. Šumskas, R. Čiegis, *Finite volume ADI scheme for hybrid dimension heat conduction problems set in a cross-shaped domain*, Lith. Math. J., (accepted), 2022.

Questions

Questions

Ačiū.